On improved crack tip plastic zone estimates based on T-stress and on complete stress fields

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ABSTRACT
Cracked ductile structures yield locally to form a plastic zone \( (pz) \) around their crack tips, which size and shape controls their structural behaviour. Classical \( pz \) estimates are based solely on stress intensity factors (SIF), but their precision is limited to very low \( \sigma_n/\sigma_Y \) nominal stress to yield strength ratios. \( T \)-stresses are frequently used to correct SIF-based \( pz \) estimates, but both SIF and SIF plus \( T \)-stress \( pz \) estimates are based on truncated linear elastic (LE) stress fields that do not satisfy boundary conditions. Using Griffith’s plate complete LE stress field to avoid such truncated \( pz \) estimates, the influence of its Williams’ series terms on \( pz \) estimation is evaluated, showing that \( T \)-stress improvements are limited to medium \( \sigma_n/\sigma_Y \) values. Then, corrections are proposed to introduce equilibrium requirements into LE \( pz \) estimates. Finally, these improved estimates are compared with \( pz \) calculated numerically by an elastic–plastic finite element analysis.

Keywords crack tip plastic zone estimates; crack tip stress fields; elastic–plastic fracture mechanics; fracture toughness; \( T \)-stress.

NOMENCLATURE

\( a \) = crack size
\( CTOD \) = crack tip opening displacement
\( E \) = Young’s modulus
\( K_i \) = stress intensity factor in mode I
\( K_t \) = stress concentration factor
\( pl - \varepsilon \) = plane strain
\( pl - \sigma \) = plane stress
\( pz \) = plastic zone
\( SIF \) = stress intensity factor
\( S_Y \) = yield strength
\( v \) = Poisson’s coefficient
\( \sigma_n \) = nominal stress
\( \sigma_n/\sigma_Y \) = nominal stress to yield strength ratio

INTRODUCTION
Most linear elastic fracture mechanics (LEFM) analysis and design routines are based entirely on stress intensity factors (SIF), a desirable and useful feature for many practical applications. However, it is well known that SIF alone do not accurately model some crack problems. For example, the idealized linear elastic (LE) stress field \( \sigma_\theta \) in a Griffith’s plate loaded in mode I by a nominal stress \( \sigma_n \), induced by its SIF \( K_i = \sigma_n \sqrt{\pi a} \), where \( 2a \) is the crack size, does not satisfy its boundary conditions far from the crack tips. Indeed, \( \sigma_\theta(r, \theta) = \left[K_i/\sqrt{2\pi r}\right] \cdot f_\theta(\theta) \Rightarrow \sigma_\theta(r \to \infty, \theta = 0) = 0 \), instead of \( \sigma_n \) \( \sigma(\text{and } \theta = 0) = \sigma_n \) as it should be, where \( r \) is the distance from the crack tip, \( \theta \) is the angle measured from the crack plane and \( f_\theta(\theta) \) are Irwin’s \( \theta \) functions. Moreover, SIF-based LE analyses obviously cannot be used to describe stresses and strains inside the plastic zones \( pz(\theta) \) that always form around real crack tips either. However, for engineering design purposes, \( pz(\theta) \) are traditionally estimated from such simplified analyses,
assuming that they depend only on SIF values. Indeed, equating the Mises stress obtained from $\sigma_y$ to the yield strength $S_Y$, the simplest mode I elastic–plastic (EP) boundary estimations in plane stress ($pl - \sigma$) and in plane strain ($pl - \varepsilon$) are expressed by:

$$pz(\theta)_{pl-\sigma} = \left(\frac{K_I^2}{2\pi S_Y^2}\right) \cos(\theta/2)^2 [1 + 3\sin(\theta/2)^2],$$  \hspace{1cm} (1)

$$pz(\theta)_{pl-\varepsilon} = \left(\frac{K_I^2}{2\pi S_Y^2}\right) \cos(\theta/2)^2 [(1 - 2\nu)^2 + 3\sin(\theta/2)^2],$$  \hspace{1cm} (2)

in which $\nu$ is the Poisson's coefficient.

According to these classical estimates, the $pz(\theta)$ size directly ahead of crack tips in $pl - \sigma$, which is the reference used here to normalize $pz(\theta)$ plots, should be $pz(0)_{pl-\sigma} = pz_0 = (1/2\pi)(K_I/S_Y)^2$. However, stress fields based solely on SIF are exact only at $r \rightarrow 0$, just where the assumed LE behaviour makes no sense. Consequently, such traditional $pz(\theta)$ estimates are reasonable only for very low SIF values, associated with nominal stress to yield strength ratios $\sigma_y/S_Y$ much smaller than those used in most practical design applications. Nevertheless, the errors introduced by this crude approximation are frequently ignored in mechanical design and analysis, even when dealing with crack problems associated with the higher $\sigma_y/S_Y$ ratios found in practice. Ignoring such errors may cause mistaken decisions, for example evaluating which component dimensions can be treated as $pl - \sigma$ or $pl - \varepsilon$ dominated for design purposes; estimating the sizes of overload-affected zones in fatigue crack propagation under variable amplitude loads; analysing how much material should be removed when repairing a fatigue crack, etc. Moreover, before proceeding, it is important to emphasize that idealized EP singular $pz(\theta)$ estimates, such as those generated by the Hutchinson, Rice, and Rosengren (HRR) field, do not solve this problem either, because they also do not satisfy stress boundary conditions far from the crack tips.

Because relatively high $\sigma_y/S_Y$ ratios are often used in structural designs nowadays, it is worthwhile to evaluate the effect of $\sigma_y/S_Y$ on $pz(\theta)$, instead of simply neglecting it. In fact, modern engineering structures are typically designed with yield safety factors $1.25 < \phi_Y < 3 \Rightarrow 0.33 < \sigma_y/S_Y < 0.8$. However, it must be pointed out that such high $\sigma_y/S_Y$ ratios do not preclude LE crack analysis, because they can many times be associated with $pz_0$ values that are small compared with the residual ligament size $rl = w - \sigma$, where $w$ is the cracked component width, particularly in large structures. In this way, their stress fields remain predominantly LE and can be well analysed as so. Indeed, note that because those typical $\phi_Y$ are associated to $0.32 < pz_0/rl < 0.06$, except for very deep cracks, their $pz_0$ size does in fact characterize a dominant LE stress field.

in $rl$. Therefore, to explore how improved LE estimates can be generated for $pz(\theta)$ sizes and shapes may be an interesting and profitable exercise, because more precise calculations for them can only be obtained by non-trivial numerical models that are far too expensive for most practical applications.

For the Griffith plate, a crude engineering LE estimate for the $\sigma_y/S_Y$ effect can be made by forcing the $\sigma_y$ stress component to obey the far field boundary condition, $\sigma_{yy}(x \rightarrow \infty, y = 0) = \sigma_y$, by adding a constant $\sigma_y$ stress to the Williams' (or Irwin) stress field to obtain

$$\sigma(\theta)_{f_{M,pl-\sigma}} = \left[(\kappa \cdot f_x)^2 + (\kappa \cdot f_y + \sigma_y)^2 - (\kappa \cdot f_x)^2 \right]^{1/2},$$  \hspace{1cm} (3)

in which $\kappa = K_I/\sqrt{2\pi r}$, and $f_x(\theta)$, $f_y(\theta)$ and $f_{yy}(\theta)$ are the mode I $\theta$ functions associated with $\sigma_{xx}, \sigma_{yy}$ and $\tau_{yx}$. The somewhat heavy notation $\sigma(\theta)_{f_{M,pl-\sigma}}$ emphasizes that Eq. (3) expresses the resulting Mises LE stress distribution around the crack tip in the limit 2D case of $pl - \sigma$ conditions, considering the $K_I$-induced stress field and the additional $\sigma_y$ constant term (arbitrarily) added to its $\sigma_{yy}$ component. A similar equation can be easily generated for $pl - \varepsilon$. The corresponding simplistic estimates for the $pz(\theta)$ border obtained by forcing $\sigma_{yy}(\theta) = S_Y$ for the Griffith's plate, shown in Fig. 1, indicate that the $\sigma_y/S_Y$ ratio may indeed significantly affect $pz(\theta)$ size and shape.

In other words, despite recognizing that adding a constant $\sigma_y$ term to the $K_I$-induced $\sigma_{yy}$ stress component is certainly not a sound approach, its simplistic $pz(\theta)$ LE estimates shown in Fig. 1 obey the far field boundary condition for the Griffith plate, thus have advantages over the even more simplistic traditional estimates expressed by Eqs (1) and (2). Moreover, in this sense they point out that their dependence on $\sigma_y/S_Y$ should be further explored, as follows in this paper.

Successively refined LE estimates are thus generated in the following sections. First, the complete LE stress field for the Griffith's plate is obtained from the equivalent Inglis plate and from its Westergaard stress function. Then, this stress field is used to estimate $pz(\theta)$ for various $\sigma_y/S_Y$ ratios. Next, the same complete LE stress field is obtained by a Williams’ series expansion, to study the effect of its number of terms on $pz(\theta)$ estimation and to show that $T$-stress improvements are limited to medium $\sigma_y/S_Y$ ratios, not compatible with the loads used in high-performance structures. In the sequence, four types of corrections are proposed to consider equilibrium effects neglected when estimating $pz(\theta)$ by forcing $\sigma_{yy}(\theta) = S_Y$. Finally, such improved LE estimates for $pz(\theta)$ are compared with plastic

zone boundaries obtained by a detailed EP analysis of the Griffith’s plate.

PLASTIC ZONE ESTIMATES BASED ON COMPLETE LE STRESS FIELDS

Plastic zone boundaries estimated by the equivalent Inglis plate

An improved estimate for the $\sigma_{pl}/S_Y$ effect on $pz(\theta)$ for the Griffith’s plate can be obtained from the LE stress field of the equivalent Inglis’ plate loaded in mode I, assuming it has a sharp elliptical notch with major semi-axis $a$ normal to $\sigma_x$ and minor semi-axis $b \ll a$. This way of modelling crack problems is interesting, because no crack can have a zero radius notch tip under load, simply because no real material can support singular stresses and strains. By assuming $x = \varepsilon \cdot \cosh(\alpha) \cdot \cos(\beta)$ and $y = \varepsilon \cdot \sinh(\alpha) \cdot \sin(\beta)$, the sharp notch used to simulate the blunted crack can be described in elliptical coordinates $(\alpha, \beta)$ by $\alpha = \alpha_0$, where $a = \varepsilon \cdot \cosh(\alpha_0)$, $b = \varepsilon \cdot \sinh(\alpha_0)$ and $\varepsilon = a/cos(\alpha_0)$. The general LE stress field in Inglis’ plates is given by a series too long to be reproduced here. However, if the sharp notch in mode I has tiny, but finite, tip radius $\rho = b^2/a = CTOD/2 \approx 2K_{Ic}^2/\pi S_Y E$, in which $CTOD$ is the crack tip opening displacement, $E = E$ in $pl - \sigma$, $E' = E/(1 - v^2)$ in $pl - \varepsilon$, $E$ is Young’s modulus and $v$ is Poisson’s coefficient, then its stress concentration factor $K_I = 1 + 2a/b$ is given by

$$K_I = 1 + 2a/b = 1 + 2 \sqrt{a \pi E S_Y / 2\sigma_a \alpha} \Rightarrow a \pi E' S_Y / 2\sigma_n = E' \phi_y / 2\sigma_n.$$ (4)

Using an $a/b$ ratio for the notch shape that simulates a blunt crack with $\alpha_0 = \tan^{-1}(b/a)$, the LE stress field in the Inglis plate can be calculated. Finally, considering the stress field describe in elliptical coordinates by $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$ and, in the $pl - \varepsilon$ case, also by $\sigma_{zz} = v(\sigma_{xx} + \sigma_{yy})$, the resulting Mises stress can be used to estimate the Inglis’ plastic zones by numerically solving Eqs (5) and (6) for $|\theta| \leq \pi$ :

$$\sigma_{M,pl-\sigma}^{log} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - 2\sigma_{xx}\sigma_{yy} + 2\tau_{xy}^2} = S_Y.$$ (5)

$$\sigma_{M,pl-\varepsilon}^{log} = \sqrt{0.5[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{xx} - \sigma_{yy})^2] + 2\tau_{xx}^2} = S_Y.$$ (6)

The results of these estimates are shown in Fig. 2. Therefore, the nominal stress to yield strength $\sigma_{pl}/S_Y$ ratio influence on Griffith’s plate $pz(\theta)$ estimated from the exact LE stress field of its equivalent Inglis’ plate, although a little smaller than that estimated by the simplistic approximation used to generate Fig. 1 plots, is indeed

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Fig. 1 Mode I $pz(\theta)$ roughly estimated for the Griffith’s plate by adding a constant $\sigma_{yy} = \sigma_{xx}$ stress to the $K_I$-based LE stress field and then finding $\sigma_{yy} = K_I + \sigma_{xx} = S_Y$ for (a) $pl - \sigma$ and (b) $pl - \varepsilon$ limit conditions, to evaluate the $\sigma_{pl}/S_Y$ effect. Note the different scales used in $pl - \sigma$ and $pl - \varepsilon$ plots.
significant. Hence, it should not be neglected in practical applications. Note that to use Inglis stress field to simulate the exact LE stress field for the Griffith’s plate, modelling its blunt crack as an elliptical sharp notch of tip radius \( \rho = \text{CTOD}/2 \), is a very reasonable hypothesis, because ideal cracks should open their tips by CTOD under load. Nevertheless, to avoid any doubts about this model soundness, it is interesting to use an alternative approach to confirm the importance of the \( \sigma_u/S_Y \) ratio in \( \Delta z(\theta) \) estimations, modelling the crack by the same Westergaard stress function used by Irwin to calculate the Griffith’s plate SIF \( K_I \). This is done in the following section.

**Plastic zones estimated using Westergaard stress functions**

The Westergaard stress functions \( Z(z) \) provide an alternative way to rigorously estimate \( \Delta z(\theta) \) from LE stress fields. However, as the EP boundary is not adjacent to the crack tip, the complete stress field generated from \( Z(z) \) must be used in such a calculation. This can be demonstrated by revisiting Irwin’s classical solution for the Griffith’s plate loaded in mode I. Thus, if \((x, y)\) and \((r, \theta)\) are Cartesian and polar coordinates centred at the crack tip, \(i = \sqrt{-1}\), and \(z = r \cdot \exp(i\theta) = x + iy\) is a complex variable, Irwin’s solution is obtained from

\[
Z(z) = (2\sigma_u)/\sqrt{z^2 - a^2} \Rightarrow Z'(z) = dZ(z)/dz = (-a^2\sigma_u)/(z^2 - a^2)^{1/2},
\]

where \( a = \text{tip radius}\), \( \rho = \text{CTOD}/2 \) and \( a \gg \rho \) in Griffith’s plate loaded in mode I. Thus, if \((x, y)\) and \((r, \theta)\) are Cartesian and polar coordinates centred at the crack tip, \(i = \sqrt{-1}\), and \(z = r \cdot \exp(i\theta) = x + iy\) is a complex variable, Irwin’s solution is obtained from

\[
\begin{align*}
\sigma_{xx} &= \frac{\text{Re}(Z(z)) - y\text{Im}(Z'(z))}{\sqrt{z^2 - a^2}} \\
\sigma_{yy} &= \frac{\text{Re}(Z(z)) + y\text{Im}(Z'(z))}{\sqrt{z^2 - a^2}} \\
\sigma_{xy} &= -\frac{y\text{Re}(Z'(z))}{\sqrt{z^2 - a^2}}
\end{align*}
\]

To solve the mode I problem from \( Z(z) \), a constant term \( \sigma_u \) is added to \( \sigma_{xy} \) to force \( \sigma_{xy}(\infty) = 0 \) in Griffith’s plate, an adequate mathematical trick, because a constant stress in the \( x \) direction does not affect the stress concentration effect introduced by the crack tip. However, the resulting \( \sigma_{yy} \) stress is usually truncated to generate its well-known SIF \( K_I = \sigma\sqrt{\pi a} \). This is normally a highly desirable feature, but the stress field truncation required to obtain it is of no use for estimating \( \Delta z(\theta) \), because it neglects the \( \sigma_u/S_Y \) effect. The stress truncation is obtained by writing

\[
\sigma_{yy}(\theta = 0) = \sigma_u(x + a)/[(x + a)^2 - a^2]^{1/2} \leq \sigma_u a/\sqrt{2ax} = K_I/\sqrt{2\pi r} \quad (if \ x < \ a),
\]

in which \( 2a \) is the crack length perpendicular to the nominal stress \( \sigma_{yy}(r \to \infty) = \sigma_u \).

As Eq. (9) formally yields \( \sigma_{yy}(\theta = 0) = K_I/\sqrt{2\pi r} = 0 \) only if \( r \to \infty \), this classical approximation is useful to obtain \( K_I \), as Irwin did. But it obviously cannot be used to study the \( \sigma_u/S_Y \) influence on \( \Delta z(\theta) \). However, this task can be fulfilled by, first, calculating the complete stress field generated from \( Z \) and \( Z' \) to obtain the resulting Mises (or Tresca, for that matter) stress and, then, by equating this stress to \( S_Y \) to obtain the required \( \Delta z(\theta) \). This calculation results in Eq. (10), which is explicitly written here to demonstrate how the Westergaard stress.
function can generate the complete LE stress field for the Griffith’s plate

\[
\begin{align*}
&\text{Re}\left(\frac{(a + r \cos(\theta) + ir \sin(\theta)) \sigma_n}{\sqrt{(a + r \cos(\theta) + ir \sin(\theta))^2 - a^2}}\right) \\
&- y \text{Im}\left(\frac{-a^2 \sigma_n}{\sqrt{(a + r \cos(\theta) + ir \sin(\theta))^2 - a^2}}\right)^2 - \sigma_n \\
&+ \left[\text{Re}\left(\frac{(a + r \cos(\theta) + ir \sin(\theta)) \sigma_n}{\sqrt{(a + r \cos(\theta) + ir \sin(\theta))^2 - a^2}}\right)\right]^2 \\
&+ \left[y \text{Im}\left(\frac{-a^2 \sigma_n}{\sqrt{(a + r \cos(\theta) + ir \sin(\theta))^2 - a^2}}\right)^2 - \sigma_n\right] \\
&+ y \text{Im}\left(\frac{-a^2 \sigma_n}{\sqrt{(a + r \cos(\theta) + ir \sin(\theta))^2 - a^2}}\right)^2 \right] \\
&+ 3 \left[-y \text{Re}\left(\frac{-a^2 \sigma_n}{\sqrt{(a + r \cos(\theta) + ir \sin(\theta))^2 - a^2}}\right)^{3/2}\right]^{1/2} \\
&- S_y = 0
\end{align*}
\]

The corresponding \(p_\varepsilon(\theta)\) estimations in \(pl - \sigma\) and \(pl - \varepsilon\) illustrated in Fig. 3 clearly show the importance of the \(\sigma_n/S_y\) effect, reinforcing the claim that that it should not be neglected in practical applications.

Moreover, it should be noted that the plastic zone boundaries estimated from complete Inglis and Westergaard LE stress fields, which satisfy all the boundary conditions for the Griffith’s plate loaded in mode I, namely \(\sigma_{yy}(|x| \leq a, y = 0) = \tau_{xy}(|x| \leq a, y = 0) = \sigma_{\infty} = \sigma_{\infty} = \sigma_n\), visually coincide when the sharp ellipses used in the equivalent Inglis’ plate has its minor semi-axis (instead of its tip radius) chosen as \(b = CTOD/2 = 2K/I\sqrt{\pi S_y E}\), as is shown in Fig. 4. As \(p_{\text{eq}}(\theta)\) and \(p_{\text{Williams}}(\theta)\) are obtained from completely different equations, their near coincidence seems to not be fortuitous. Therefore, the large \(\sigma_n/S_y\) effect estimated by these rigorous LE solutions really should not be neglected in practice. It is worthwhile to emphasize this point again, because it is the plastic zone size that validates most LEFM predictions used in design applications.

**Plastic zones estimated using Williams series and only the T-stress correction**

Based on photoelastic experimental results obtained by Wells and Post\(^5\) for the stress field around crack tips, Irwin proposed a long time ago to add a constant term to the stress component \(\sigma_{xx}\) parallel to the crack direction, naming it T-stress,\(^6\) to improve LEFM predictions. Later on, Larsson and Carlsson,\(^7\) while investigating the limits recommended by ASTM for SIF use, noted that the addition of the T-stress term could adjust \(p_\varepsilon(\theta)\) shapes estimated by LE analysis, approximating them to the \(p_\varepsilon(\theta)\) shapes obtained from EP finite elements (FE) numerical analyses. Following that work, the T-stress has been widely explored in the literature to model some interesting problems. In this context, it is worth mentioning the works of Rice,\(^8\) Leevers and Radon,\(^9\) Cardew et al.,\(^10\) Kfouri,\(^11\) Bilby et al.,\(^12\) Sham,\(^13\) Beţegon and Hancock,\(^14\) Du and Hancock,\(^15\) O’Dowd and Shih,\(^16\) Nakamura and Parks,\(^17\) Wang and Parks,\(^18\) Wang,\(^19\) Hancock et al.,\(^20\) Kim et al.,\(^21\) Ganti and Parks,\(^22\) Zhang et al.,\(^23\) Ramesh et al.,\(^24\) Chen and Tian\(^25\) Kang and Beom,\(^26\) Smith et al.,\(^27\) Zhao et al.,\(^28\) Chen et al.,\(^29\) Karihaloo and Xiao,\(^30\) Tan and Wang,\(^31\) and Su and Sun.\(^32\) Fett\(^33\) listed T-stress values for several geometries. The T-stress addition results in a mode I LE stress field given by

\[
\begin{align*}
\sigma_{xx} & = \frac{K_I}{\sqrt{2\pi r}} \cos(\theta/2) \left[1 - \sin(\theta/2) \sin(\theta/2)\right] \\
\sigma_{yy} & = \frac{K_I}{\sqrt{2\pi r}} \sin(\theta/2) \left[1 + \sin(\theta/2) \sin(\theta/2)\right] \\
\sigma_{xy} & = 0
\end{align*}
\]

However, because the \(K_I + T\)-stress field cannot reproduce the \(\sigma_{\infty} = \sigma_n\) boundary condition, despite its popularity it must treated as a simplification for the complete LE stress field used above for analysing the Griffith’s plate.

Fett showed that the T-stress is the Williams’ series constant or zero order term. Dumont and Lopes\(^34\) demonstrated that, to obtain \(K_I\) values using Kelvin’s solution in the Hybrid Boundary Element Method, the complete Williams series should be used to generate a suitable stress function. This series is given by

\[
\Phi = r^{\lambda+1} \times \left\{c_1 \sin[(\lambda + 1)(\theta + \pi)] + c_2 \cos[(\lambda + 1)(\theta + \pi)] + c_3 \sin[(\lambda - 1)(\theta + \pi)] + c_4 \cos[(\lambda - 1)(\theta + \pi)]\right\}
\]

where \(c_1, c_2, c_3\) and \(c_4\) are constants and \(\lambda\) is an exponent to be determined from the cracked component boundary.
Fig. 3 Mises $p_z(\theta)$ for the Griffith’s plate loaded in mode I, estimated from its complete LE stress field calculated from its Westergaard stress function for (a) $pl - \sigma$ and (b) $pl - \epsilon$ conditions.

Fig. 4 Mises $p_z(\theta)$ estimated from the complete Westergaard stress field for (a) $pl - \sigma$ and (b) $pl - \epsilon$ are visually identical to Inglis estimates if a sharp elliptical notch with $b = CTOD/2 = 2K_I/\pi S_Y F'$ instead of $\rho = CTOD/2$ is used to model the crack.

conditions. Eq. (12) can be compactly rewritten as $\Phi = r^{\lambda+1} F(\theta, \lambda)$ and, to be a stress function, it must satisfy the following expression

\[
\begin{align*}
\sigma_{rr} & = \left\{ \begin{array}{l} 
\frac{1}{r^{\lambda+1}} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} \\
\frac{1}{r^{\lambda+1}} \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} 
\end{array} \right\} \\
\sigma_{\theta \theta} & = \frac{1}{r^{\lambda+1}} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} \\
\sigma_{r \theta} & = \lambda \frac{1}{r^{\lambda+1}} \frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial \Phi}{\partial r} \\
\sigma_{\theta r} & = \frac{1}{r^{\lambda+1}} \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta}
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
\sigma_{rr} & = \frac{\partial^2 \Phi}{r} + \frac{\partial \Phi}{r} \\
\sigma_{\theta \theta} & = \frac{\partial^2 \Phi}{r} \\
\sigma_{r \theta} & = \lambda \frac{\partial^2 \Phi}{r} \\
\sigma_{\theta r} & = -\lambda \frac{\partial \Phi}{r}
\end{cases}
\]

where $F'(\theta)$ is the derivative of $F$ with respect to $\theta$. With the crack faces in the negative $x$-axis direction, their free surfaces require $\sigma_{\theta \theta}(\pm \pi) = \sigma_{r r}(\pm \pi) = 0$, then

\[
\begin{align*}
\sigma_{rr} & = r^{\lambda-1} \left\{ \begin{array}{l} 
F''(\theta) + (\lambda + 1) F(\theta) \\
\lambda (\lambda + 1) F(\theta) \\
-\lambda F'(\theta)
\end{array} \right. 
\end{align*}
\]
If \( F(\pm \pi) = F(\pm \pi) = 0 \) if \( \lambda \neq 0 \). Consequently, \( c = -c_3 \), \( c_1 = (1 - \lambda)/(1 + \lambda) - c_3 \) and, because negative \( n \) would be associated with an infinite strain energy at the crack tip, \( \sin(2\pi\lambda) = 0 \Rightarrow \lambda = n/2, n = 0, 1, 2, 3, \ldots \). The resulting LE stress field given by Eq. (13) can be rewritten in terms of \( \sigma_{xx}, \sigma_{yy} \), and \( \sigma_{xy} \), as preferred in the LEFM literature, using proper \( f_{xx}, f_{yy}, f_{xy} \)-functions for each \( i \)-th term of the Williams’ series

\[
\begin{bmatrix}
\sigma_{xx}
\
\sigma_{yy}
\
\sigma_{xy}
\end{bmatrix} = \frac{K_f}{\sqrt{2\pi r}} \begin{bmatrix}
1 - \sin(\theta/2)(3\theta/2) \\
1 + \sin(\theta/2)\sin(3\theta/2) \\
\sin(\theta/2)\sin(3\theta/2)
\end{bmatrix}
+ T\begin{bmatrix}
f_{xx}(\theta)
\
f_{yy}(\theta)
\end{bmatrix} + \kappa_2^{3/2} \begin{bmatrix}
f_{xy}(\theta)
\end{bmatrix} + \cdots
\]

(14)

Similar equations can be obtained for pure mode II and for mixed mode I-Mode II loading conditions but, because they are not required to demonstrate the \( \sigma_n/S_Y \) effect, there is no need to reproduce them here. After proper transformations, the first term of the stress field generated from the Williams series given by Eq. (14) reproduces the \( K_f \) field and the second term (in fact, the constant or zero order term, correspondent to \( \lambda = 0 \)) gives the \( T \)-stress contribution expressed in Eq. (11). Its higher order terms (corresponding to \( \lambda = 2, 3, \ldots \)) can be used to fit complete LE fields of any cracked planar component, calculated or experimentally obtained by any means. For the Griffith’s plate, the Williams series coefficients can be adjusted to its analytical solution, obtained, for example from Eq. (8). For more complex geometries, the higher order Williams’ coefficients can be fitted to numerically calculated LE stress fields, generating in this way analytical expressions for them. This task may be achieved using for example the least squares method, after defining the following vectors and matrices:

1. \([F]\) matrix, which stores in its rows the \( n \) first Williams’ \( F_n(\theta) \) terms evaluated at the points obtained from the reference stress function (for the Griffith’s plate analysed here, calculated from the complete stress field generated by its Westergaard stress function);
2. \(\{c\}\) vector, containing the coefficients of the \( n \) first Williams’ terms;
3. \(\{s_n\}\) vector, containing the reference stresses (analytically calculated here, but which may be numerically calculated or else experimentally measured in more complex cases).

According to the least squares method, the vector \(\{c\}\) must be such that

\[
(\{c\}^T[F)^T - (s_n)^T]) \times (F}[c] - (s_n)) = \min \Rightarrow \{c\} = ([F]^T[F])^{-1}[F]^T[s_n].
\]

(15)

For low loads, that induce fairly small \( \sigma_n/S_Y \) ratios, say \( \sigma_n/S_Y < 0.1 \), Figs. 2 and 3 show that just the first Williams’ term (the \( K_f \) value) is enough to reproduce reasonably well \( p_{2\pi}(\theta) \) boundaries estimated from the complete LE stress field of the Griffith plate. For medium loads, say for \( \sigma_n/S_Y < 0.4 \), this behaviour is achieved by LE stress fields generated by using two terms in Williams series, namely \( K_f \) and the \( T \)-stress, as is shown in Fig. 5.

However, when the value of \( \sigma_n/S_Y \) is increased, the addition of the \( T \)-stress only is no more sufficient to reproduce the \( p_{2\pi}(\theta) \) estimated from the complete LE stress field of the Griffith’s plate. This fact is illustrated in Fig. 6, which plots \( p_{2\pi}(\theta) \) estimates for \( \sigma_n/S_Y = 0.8 \). Six Williams terms plus the \( T \)-stress are necessary to reproduce \( p_{2\pi}(\theta) = p_{2\pi}^{W_6}(\theta) \) in \( pl - \sigma \), whereas five Williams terms plus the \( T \)-stress are needed in \( pl - \varepsilon \) in this case. The coefficients of these terms are given by

\[
\sigma_{xx}^{W_6} = \frac{0.56569}{\sqrt{\pi}} \cos(\frac{\theta}{2}) \left[ 1 - \sin(\frac{\theta}{2}) \sin(\frac{3\theta}{2}) \right] - 0.421 \sqrt{\pi}
\times \cos(\frac{\theta}{2}) \left[ \cos(\frac{\theta}{2}) - 2 \right] - 0.0768 \rho^{1/2}
\times \cos(\frac{\theta}{2}) \left[ 7 \cos(\frac{\theta}{2}) - 6 \right] + 0.0168 \rho^{3/2}
\times \cos(\frac{\theta}{2}) \left[ 36 \cos(\frac{\theta}{2}) - 45 \cos(\frac{\theta}{2})^2 + 10 \right]
- 0.00264 \rho^{7/2} \cos(\frac{\theta}{2}) \left[ -14 + 176 \cos(\frac{\theta}{2})^6 \right]
- 308 \cos(\frac{\theta}{2})^4 + 147 \cos(\frac{\theta}{2})^2
+ 0.0002539 \rho^{9/2} \cos(\frac{\theta}{2}) \left[ 1368 \cos(\frac{\theta}{2})^4 \right]
- 345 \cos(\frac{\theta}{2})^2 + 18 + 832 \cos(\frac{\theta}{2})^8
- 1872 \cos(\frac{\theta}{2})^6 - \sigma_n
\]

(16)
Fig. 5 Mises $p_2(\theta)$ estimated for the Griffith’s plate loaded in mode I from Williams’ series with two terms, which induce a truncated $K_I + T$-stress LE stress field, visually reproduce reasonably well $p_2^{Wig}(\theta)$ (estimated from the plate complete LE stress field), both (a) in $p_l - \sigma$ and (b) in $p_l - \varepsilon$, for a medium nominal stress to yield strength ratio $\sigma_n/S_Y \leq 0.4$.

Fig. 6 Mises $p_2(\theta)$ estimated in (a) $p_l - \sigma$ and (b) in $p_l - \varepsilon$ for the Griffith’s plate in mode I from the Williams series needed up to six terms $+ T$-stress to reproduce $p_2^{Wig}(\theta)$ for $\sigma_n/S_Y = 0.8$. 
\[
\sigma_{yy}^{\text{Wtg}} = \frac{0.56569}{\sqrt{r}} \cos \left(\frac{\theta}{2}\right) \left[ 1 + \sin \left(\frac{\theta}{2}\right) \sin \left(\frac{3\theta}{2}\right) \right] + 0.421 \sqrt{r} \\
\times \cos \left(\frac{\theta}{2}\right)^3 - 0.0768 r^{3/2} \cos \left(\frac{\theta}{2}\right)^3 - 0.0168 r^{1/2} \\
\times \cos \left(\frac{\theta}{2}\right)^3 \left[ 4 \cos \left(\frac{\theta}{2}\right)^2 - 5 \right] + 0.00264 r^{7/2} \\
\times \cos \left(\frac{\theta}{2}\right)^3 \left[ 35 + 48 \cos \left(\frac{\theta}{2}\right)^4 - 84 \cos \left(\frac{\theta}{2}\right)^2 \right] \\
- 0.0002539 r^{9/2} \cos \left(\frac{\theta}{2}\right)^3 \left[ 504 \cos \left(\frac{\theta}{2}\right)^2 - 105 \right] \\
+ 320 \cos \left(\frac{\theta}{2}\right)^6 - 720 \cos \left(\frac{\theta}{2}\right)^4 - 0.0843 r^{5/2} \\
\times \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)^2 \left[ 4 \cos \left(\frac{\theta}{2}\right)^2 - 3 \right] \\
+ 0.01848 r^{7/2} \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)^2 \left[ 16 \cos \left(\frac{\theta}{2}\right)^3 - 20 \cos \left(\frac{\theta}{2}\right)^2 + 5 \right] \\
- 0.002285 r^{9/2} \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)^2 \left[ 64 \cos \left(\frac{\theta}{2}\right)^6 - 112 \cos \left(\frac{\theta}{2}\right)^4 + 56 \cos \left(\frac{\theta}{2}\right)^2 - 7 \right].
\]

(17)

However, for \( \sigma_y/S_y = 0.8 \) the Mises \( pz(\theta) \) estimated for the Griffith’s plate loaded in mode I from the Williams series need at least 6 terms + \( T \)-stress to visually reproduce \( pz^{\text{Wtg}}(\theta) \) in (a) \( pl - \sigma \) and 5 terms + \( T \)-stress to do so in (b) \( pl - \varepsilon \), see Fig. 6.

Although the complete field generated, for example from the Westergaard stress function is the correct LE solution for the Griffith’s plate, it is important to remember that the stress distribution inside \( pz(\theta) \) cannot be consistent with its (assumed) singular LE stress field. Indeed, the LE hypothesis inevitably leads to underestimated \( pz(\theta) \) boundaries. As a first approximation, these stresses can be limited to the yield strength \( S_y \), neglecting strain-hardening effects inside \( pz(\theta) \). In this way, four alternative models for compensating the stress truncation inside the plastic zones by forcing the plate to satisfy equilibrium requirements are discussed in the following section.

**LE Plastic Zones Estimates Modified to Satisfy Equilibrium Conditions**

The first correction to force a LE \( pz \) estimate to satisfy equilibrium conditions was proposed a long time ago\(^2\): analysing Griffith’s plate, Irwin corrected the plastic zone size in the direction parallel to the crack plane, shifting the originally singular LE \( \sigma_y(r) \) stress component generated by \( K_I \) to compensate for its yield-induced truncation inside \( pz(r) \). If strain hardening is neglected, it is well known that such an approach doubles the \( pz(r) \) size ahead of the crack tip. Lacking a sound analytical solution for this EP stress analysis problem, some proposals to estimate the equilibrium requirement influence on the entire size and shape of the EP frontier \( pz(\theta) \) ahead of the crack tip are described below.

**Correction proposed by Rodriguez et al.\(^3\)**

This correction compensates for the \( \sigma_y(r, \theta) \) component truncation by assuming that

\[
pz(\theta)_{M}^{\text{Wtg+eq}} = \frac{\int_{0}^{\phi_{\theta}} \sigma_y(r, \theta) dr}{\sigma_y \left( \pz(\theta)_{M}^{\text{Wtg}} \right)}. \tag{19}
\]

This equation may be seen as an attempt to generalize Irwin’s classical idea to compensate for the lack of equilibrium induced by the yield-induced stress truncation inside the entire \( pz(\theta) \) region. It basically forces the equilibrium of the net vertical forces that cannot exist within the plastic zone because \( \sigma_y(r, \theta) \) is limited by the yield stress inside it, if strain-hardening is neglected.\(^4\) Besides generalising the equilibrium requirement of vertical forces along any \( \theta \)-direction, the most important difference between Eq. (19) and Irwin’s model is that the former is based on the complete Westergaard stress function (thus the notation \( W_{tg} + eq \sigma_y \) used to describe it), whereas the latter considers a truncated stress field based solely on the SIF.

**Correction based on a constant \( pz(\theta) \) increment in each \( \theta \)-direction**

The constant \( pz(\theta) \) increment along each radius defined by a \( \theta \)-direction is obtained by

\[
pz(\theta)_{M}^{\text{Wtg+eq}} = \pz(\theta)_{M}^{\text{Wtg}} + \kappa, \tag{20}
\]

in which \( \kappa = pz(\theta = 0)_{M}^{\text{Wtg+eq}} - pz(\theta = 0)_{M}^{\text{Wtg}} \). This \( \kappa \) constant has the same equilibrium rationale as the previous correction, but it is based on a yield-induced stress.
truncation compensation inside $p_z$ along the crack direction $\theta = 0$. For other radial directions, the same length correction is adopted, inspired by the idea of a constant $T$-stress correction.

**Correction based on the equivalent Mises stress**

This correction compensates yield-induced stress truncation inside $p_z(\theta)$ by forcing

$$p_z(\theta)_{M}^{WEG + eqM} = \int_0^r \frac{p_z(\theta)_M}{S_T} dr,$$  \hspace{1cm} (21)

Because Eq. (19) only takes into account the effect of the $\sigma_{yy}$ stress component for $\theta \neq 0$, the equilibrium correction of yield-induced stress truncation inside $p_z(\theta)$ by the Mises stress $\sigma_{M}(r, \theta)$ may be seen as a reasonable alternative, because it is affected by all stress components. Besides, it can be extended to any type of loading.

**Correction using the vertical traction component**

This correction uses the vertical traction $t_r$ component to compensate for the yield-induced stress truncation inside $p_z(\theta)$:

$$p_z(\theta)_{M}^{WEG + eqTr} = \int_0^r \frac{t_r(r, \theta) dr}{t_r(\theta)_{M}^{WEG}},$$  \hspace{1cm} (22)

where $t_r$ is determined by

$$\begin{bmatrix} t_x(r, \theta) \\ t_y(r, \theta) \end{bmatrix} = \begin{bmatrix} \sigma_x(r, \theta) & \tau_{xy}(r, \theta) \\ \tau_{xy}(r, \theta) & \sigma_y(r, \theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$  \hspace{1cm} (23)

Once again, this correction only produces an exact equilibrium for $\theta = 0$. However, by considering the vertical traction component, the equilibrium may be seen as resulting from a free body diagram obtained by sectioning the model along any $\theta$-direction.

Figs. 7 and 8 compare the equilibrium correction alternatives described earlier, by showing the (slight) difference between their $p_z(\theta)$ estimates for plane stress (Fig. 7) and plane strain (Fig. 8). These figures also depict plastic zones obtained by truncated SIF, SIF plus $T$-stress and complete LE stress fields, which do not satisfy equilibrium requirements.

Note that the $K_I + T$-stress $p_z(\theta)$ are significantly different from the $p_z(\theta)$ corrected for equilibrium. Note also that all $p_z(\theta)$ estimates made until now were obtained from LE analyses. However, no matter how elaborated, these approaches can have only a qualitative character, because the EP $p_z(\theta)$ estimation problem is intrinsically incremental. Therefore, the next section shows $p_z(\theta)$ estimates obtained from an incremental non-linear analysis using the Finite Element Method, to compare them with the LE estimates developed earlier.

**PLASTIC ZONES ESTIMATES FROM EP FEM ANALYSIS**

All estimates for $p_z(\theta)$ boundaries presented so far were obtained from truncated LE stress fields considering $\sigma_M(\theta) = S_T$, or at most have estimated equilibrium effects induced by such a truncation. However, the $p_z(\theta)$ estimation problem cannot be modelled well by such global approaches. Indeed, each small increment in the stress level causes yielding in the vicinity of the crack tip and redistributes the local stresses. Consequently, the resulting plastic zones grows incrementally while the load is applied to the structure. Therefore, the present section analyses the Griffith’s plate properly considering this EP process to estimate $p_z^{EP}(\theta)$ by incremental EP FE analyses. These analyses neglect strain-hardening effects, to be consistent with the previous LE estimates. Their parameters, for example the maximum number of equilibrium iterations (eq iter = 2000) and the load increments (inc load = 0.1 of full load), were adjusted according to ANSYS documentation. Fig. 9 compares the calculated $p_z(\theta)$ with the equilibrium-corrected LE $p_z(\theta)$ estimates that use $\sigma_{yy}$ component and $\sigma_{M}_{eq}$, and with the $p_z(\theta)$ obtained from truncated SIF and SIF plus $T$-stress fields for $\sigma = 0.4, 0.6, 0.7$ and 0.8.

Note how the shapes of LE plastic zone estimates are quite different from those calculated incrementally by a proper EP analyses. This is an indication that not even complete LE fields can generate satisfactory $p_z(\theta)$ estimates for some design purposes under high $\sigma_{yy}/S_T$ ratios. If precise plastic zone evaluations are needed in such cases, it may be indispensable to calculate them by proper EP numerical models. However, as such models are complex and expensive, simplified LE estimates will probably remain unavoidable for the foreseeable future. Moreover, because EP models are not foolproof, experimental verification may be needed to guarantee their predictions, in particular when strain-hardening effects cannot be neglected.

LE estimates based on truncated stress fields generated by the $K_I + T$ approximation lead to reasonable values for the maxima $p_z(\theta)$ sizes in $p_l - \sigma$ in this Griffith plate, but severely over-estimated the $p_z(\theta)$ boundaries calculated by the EP model in $p_l - \varepsilon$ for high $\sigma_{yy}/S_T$ loads. Equilibrium-corrected LE estimates based on the complete LE stress field are less bad in
this case. However, such behaviour cannot be extrapolated for other cases. Anyway, it can be certainly stated that $p_z(\theta)$ estimates based solely on $K_I$ should not be assumed safe for modelling purposes, let alone for design applications.

**CONCLUSIONS**

To evaluate the accuracy of plastic zones size and shape estimates frequently used for mechanical design and structural integrity evaluations of cracked components, estimates for $p_z(\theta)$ ahead of Griffith’s plate crack tips
obtained from five LE stress fields are compared to that determined for its EP stress field calculated by FE. The first LE field is based solely on its SIF. The second is obtained from its SIF plus its $T$-stress, which is its Williams’ series constant or zero order term. The third is based on the Griffith’s plate complete LE stress field generated by its Westergaard stress function, the correct analytical solution for this stress analysis problem. The fourth is based on the complete solution for the LE stress field of an elliptical sharp notch in the equivalent Inglis’ plate, that is, the notch that simulates the opened Griffith crack by an elliptical hole with major axis equal to the crack length and minor semi-axis equal to $b = 2K_I^2/\pi S_Y E'$. The last LE stress field, based on the complete Williams series expansion for the Griffith’s plate, is used to evaluate the number of its terms needed to fit well $p_z(\theta)$ estimates generated by the complete Westergaard stress field. The accuracy of classical $p_z(\theta)$ estimates based solely on SIF are limited to very low $\sigma_n/S_Y$ ratios, and estimates based on SIF plus $T$-stress, frequently assumed to be an appropriate solution for this problem, do not solve it either. These approximate LE stress fields do not satisfy boundary conditions, in particular the nominal stresses far from the crack tip. On the other hand, the complete stress field generated from the Westergaard stress function satisfies those conditions, because it is the correct LE solution for the Griffith’s plate. Its estimates for $p_z(\theta)$...
Fig. 10 Comparison of $p_z(\theta)$ calculated by EP incremental FE analyses for the Griffith’s plate loaded in mode I in $pl - \varepsilon$ with $p_z(\theta)$ estimated from complete LE stress fields corrected for equilibrium using the $\sigma_{\text{Mises}}$ component (which gives estimates similar to equilibrium corrections based on $\sigma_{yy}$ and on $t_y$) and with truncated LE estimates based only on the $T$-stress, $p_z^{K_I+T}(\theta)$.

can be reproduced by solving the equivalent Inglis’ plate. The Williams series can also reproduce such estimates, if a proper number of terms is used, which depends on the $\sigma_n/S_Y$ ratio. This approach demonstrates that $p_z(\theta)$ estimates based on SIF plus $T$-stress do not reproduce the estimates based on LE complete stress field for high $\sigma_n/S_Y$ values.

Nevertheless, none of the $p_z$ estimates based on LE stress fields satisfy equilibrium conditions, because their singular (or very high, in the Inglis case) predicted stresses at the crack tip must be limited inside the plastic zone. To cope with this problem, four alternative models for compensating the stress truncation neglecting strain-hardening effects inside the plastic zone are considered, to force the plate to satisfy equilibrium conditions. Such equilibrium-corrected LE estimates have been compared with $p_z(\theta)$ incrementally calculated using an EP FE code, also neglecting strain hardening to maintain compatibility with the LE estimates. Whereas the shape of equilibrium-corrected $p_z(\theta)$ LE estimates are quite different from the EP ones, their maximum size for high $\sigma_n/S_Y$ levels is reasonably well reproduced both in $pl - \sigma$ and $pl - \varepsilon$, contrary to what happens with estimates based on truncated LE stress fields generated from SIF plus $T$-stress, which severely overestimate the plastic zones in $pl - \varepsilon$ in this Griffith’s plate case.
REFERENCES