The dynamics of the Jouanolou foliation

Cesar Camacho
Luiz Henrique de Figueiredo

IMPA–Instituto de Matemática Pura e Aplicada, Brazil

to appear in *Ergodic Theory and Dynamical Systems*

Introduction

PROBLEM (Camacho-Lins-Sad, 1988)

Are there foliations of $\mathbf{C}\mathrm{P}(2)$ *with nontrivial minimal sets?*

Negative answer $\approx$ Holomorphic Poincaré-Bendixson theorem

A *minimal set* of a foliation is an invariant, closed, nonempty subset of $\mathbf{C}\mathrm{P}(2)$ such that it is minimal with these three properties.

A minimal set is *nontrivial* if it is not a singular point.

Properties of nontrivial minimal sets

If $\mathcal{M}$ is a minimal sets for a foliation $\mathcal{F}$ in $\mathrm{CP}(2)$, then:

- $\mathcal{M}$ is unique;
- there is no $\mathcal{F}$-invariant transverse measure supported on $\mathcal{M}$, and so all the leaves in $\mathcal{M}$ have exponential growth;
- $\mathcal{M}$ intersects every one-dimensional algebraic subset of $\mathrm{CP}(2)$, and thus $\mathcal{F}$ cannot have algebraic leaves.
  Moreover, there are leaves in $\mathcal{M}$ with nontrivial linear holonomy

The Jouanolou foliation

Example of Darboux, studied by Jouanolou (1979):

$$\mathcal{D}_k : \quad \begin{vmatrix} dX & dY & dZ \\ X & Y & Z \\ Y^k & Z^k & X^k \end{vmatrix} = 0$$

- $\mathcal{D}_k$ has degree degree k and $N = k^2 + k + 1$ singular points.
- $\mathcal{D}_k$ was used to prove that the set of foliations in $\mathrm{CP}(2)$ of degree k that do not admit algebraic leaves is open and dense in the space of all foliations of degree k (Jouanolou, 1979; Lins, 1988).

QUESTION

Does $\mathcal{D}_k$ admits a nontrivial minimal set?

ANSWER

No, for $k = 2, 3, 4, 5$. Possibly not for all k.

Strategy of the proof

1. Exploit the symmetries associated to $\mathcal{D}_k$ to find small, thin regions of $\mathbf{C}\mathrm{P}(2)$ that every minimal set of $\mathcal{D}_k$ must cross. We prove that two small sectors of angle π/N of the unit disk on the coordinate planes $\mathbf{C} \times 0$ and $0 \times \mathbf{C}$ suffice.
2. Find a sphere centered at the real singular point $(1, 1)$ that is transversal to $\mathcal{D}_k$. Then, every orbit that enters this sphere must accumulate on $(1, 1)$.
3. Show that all orbits starting in the two sectors enter this transversal sphere in finite time. It is enough to consider the *real* flow.

We give *reliable computational proofs* of steps 2 and 3 using on interval arithmetic.

The symmetries of $\mathcal{D}_k$

Affine expression:

$$\dot{x} = y^k - x^{k+1}$$
$$\dot{y} = 1 - x^k y$$

$\text{sing}(\mathcal{D}_k) = \left\{ (\zeta^j, \zeta^{-jk}) : j = 0, \ldots, N-1 \right\}$

$N = k^2 + k + 1$

$\zeta = N$-th root of unity

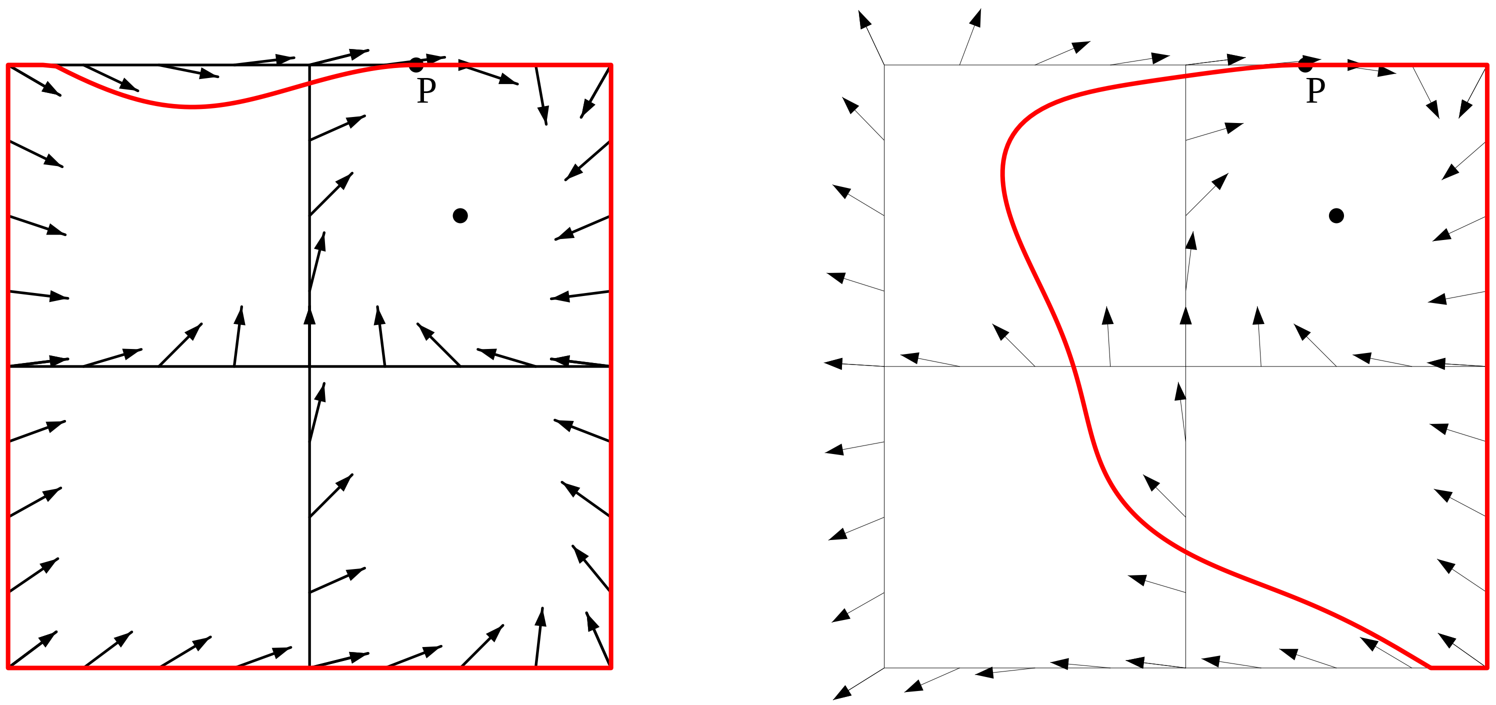
- $\mathcal{D}_k$ invariant under
  $S(X, Y, Z) = (Y, Z, X)$
  $T(X, Y, Z) = (\zeta X, \zeta^{-k} Y, Z)$

- The leaf that contains the origin accumulates on all singularities of $\mathcal{D}_k$. Because of symmetries, it is enough to consider the real phase space.

The leaf of the origin accumulates on all singularities

k even: the ω-limit set of every point in $\mathbf{R}^2$ is the singular point $(1, 1)$.

k odd: the ω-limit set of $(0, 0)$ is the singular point $(1, 1)$.



k even k odd

Locating minimal sets

- If $\mathcal{M}$ is a nontrivial minimal set of $\mathcal{D}_k$, then $\mathcal{M}$ intersects the polydisc $\left\{ (x,y) \in \mathbf{C}^2 : |x| \leq 1, |y| \leq 1 \right\}$. More precisely, $\mathcal{M}$ has a nontrivial intersection either with the disc $D_1 = \left\{ (x,0) \in \mathbf{C}^2 : |x| \leq 1 \right\}$ or with the disc $D_2 = \left\{ (0,y) \in \mathbf{C}^2 : |y| \leq 1 \right\}$
- To prove that a nontrivial minimal set for $\mathcal{D}_k$ does not exist, it is enough to show that the leaves of $\mathcal{D}_k$ passing through $D_1 \cup D_2$ accumulate on the singularities of $\mathcal{D}_k$.
- Since $\mathcal{D}_k$ is invariant by $\tau(x,y) = (\zeta x, \zeta^{-k} y)$ and also by complex conjugation, it is enough to verify this in sectors of D_1 and D_2 of angle π/N. The simplest such sectors are $S_1 = S_0 \times 0 \subseteq D_1$ and $S_2 = 0 \times S_0 \subseteq D_2$, where

$$S_0 = \left\{ z \in \mathbf{C}^2 : |z| \leq 1,\, 0 \leq \arg(z) \leq \pi/N \right\}.$$

A large transversal sphere around the real singularity

The sphere in $\mathbb{C}^2$ *of radius* R *centered at* $(1,1)$ *is transversal to* $\mathcal{D}_k$*.*

PROOF. We have to show that

$$\langle (x-1, y-1), (y^k - x^{k+1}, 1 - x^k y)\rangle_{\mathbb{C}} \neq 0,$$

for all points (x, y) on the boundary of the sphere. We show that

$$f(x_1, x_2, y_1, y_2) = \mathrm{Re}\langle (x-1, y-1), (y^k - x^{k+1}, 1 - x^k y)\rangle_{\mathbb{C}} \leq M \ll 0$$

by using interval methods for global optimization.

The computed values of R and M are:

k	2	3	4	5
R	1.02	0.57	0.36	0.26
M	–0.017	–0.0020	–0.0064	–0.0015

All orbits approach the real singularity

All orbits starting in the sectors S_1 *and* S_2 *enter the transversal sphere in finite time.*

PROOF.

- Cover each sector with a set of rectangles.
- Show that each rectangle is mapped into the transversal sphere by the real flow.

We used AWA, a program by Rudolf Lohner for the reliable solution of initial value problems. AWA is able to give a reliable bound for the location at a given time of *all* orbits starting in a given box.

More precisely, given a box B_0 and a time t_1, AWA uses interval arithmetic to compute a box B_1 such that every orbit starting in B_0 is inside B_1 at time t_1 (AWA also proves that the solution exists at t_1).

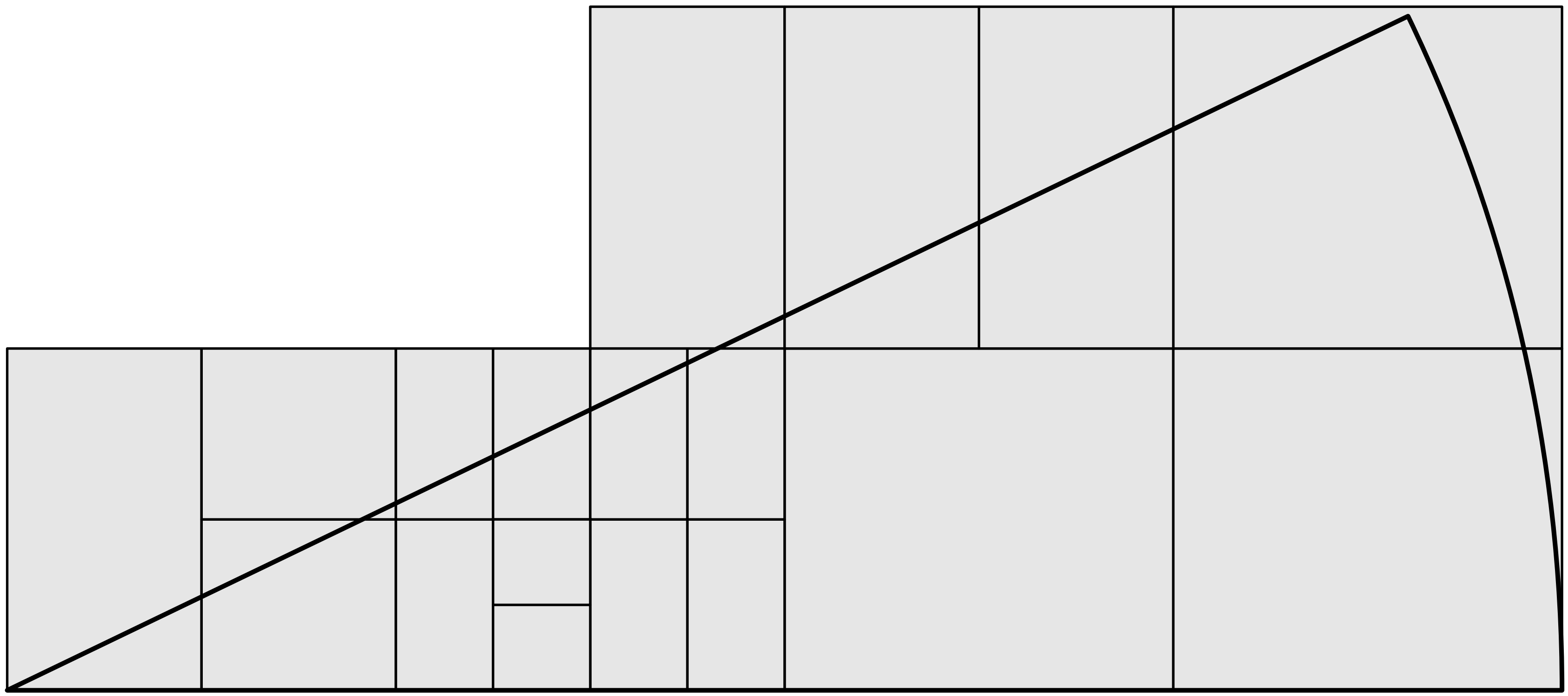
Covering a sector with rectangles

Adaptive algorithm

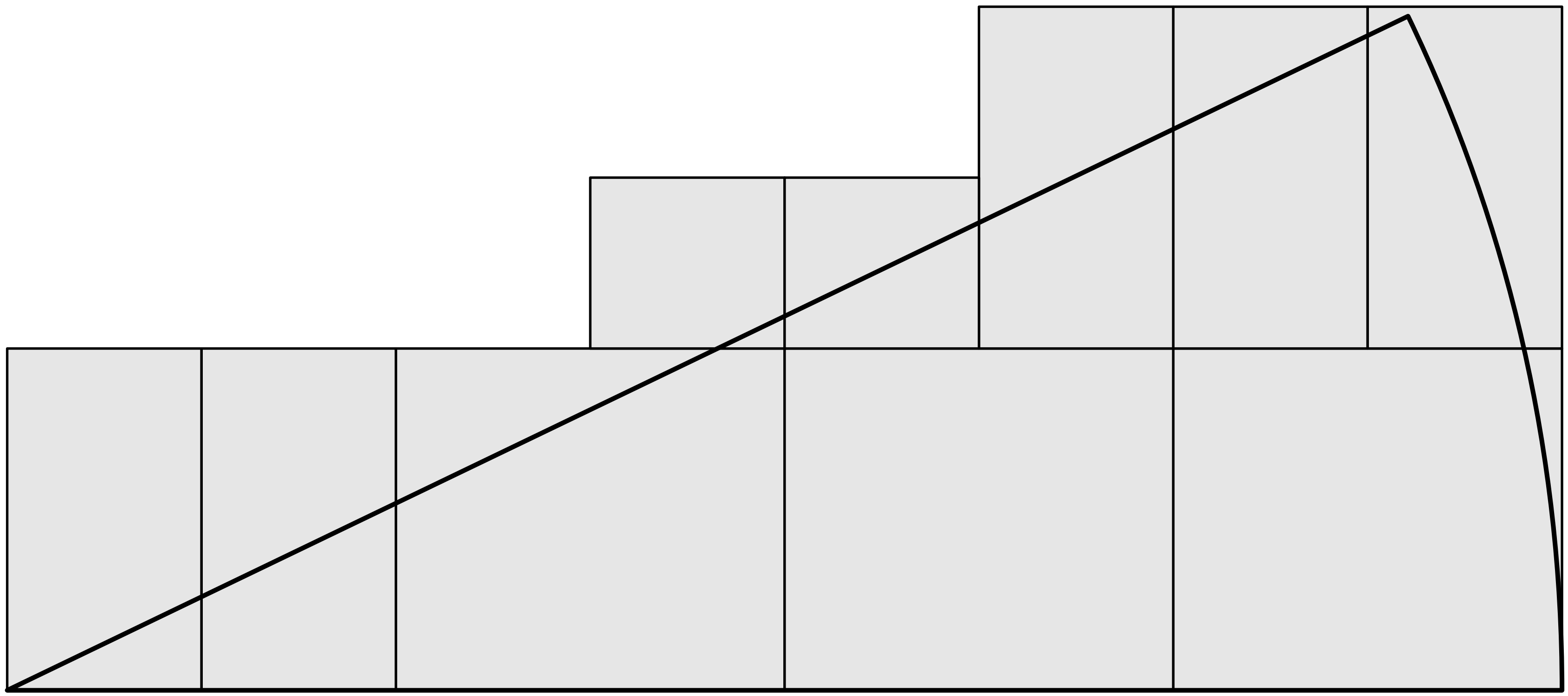
Input: Rectangle B covering sector S.
Output: List of rectangles covering S such that each rectangle has been proved by AWA to be taken into the transversal sphere.

```
𝓛 ← {B}
while 𝓛 is not empty do
    select a rectangle X from 𝓛
    if X intersects S then
        run AWA over X until X is taken into the sphere or AWA gives up
        if X was taken into the sphere then
            output X
        else
            subdivide X into X_1 and X_2
            put X_1 and X_2 into 𝓛
```

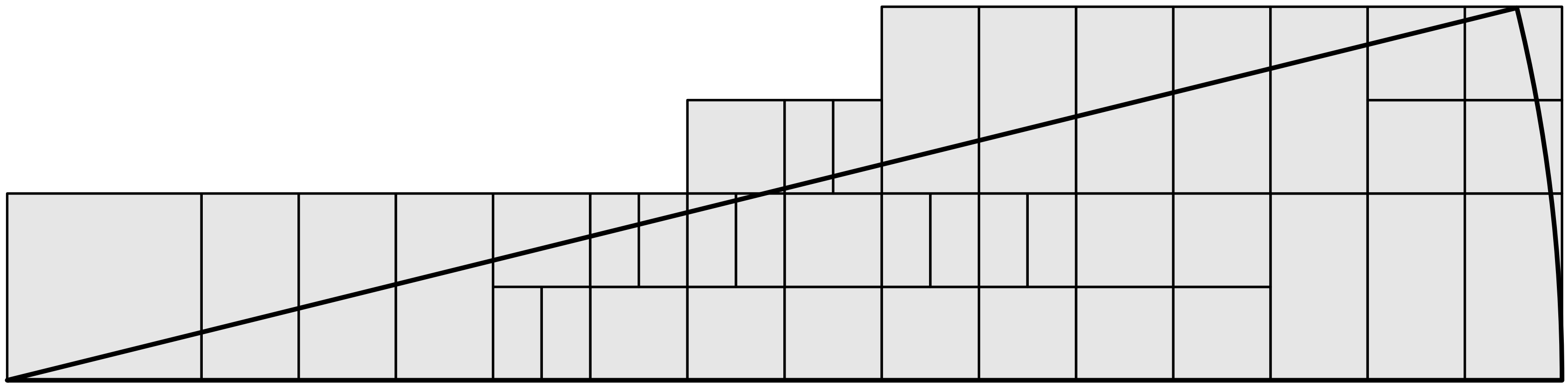
Covering for $k = 2$ (x axis)

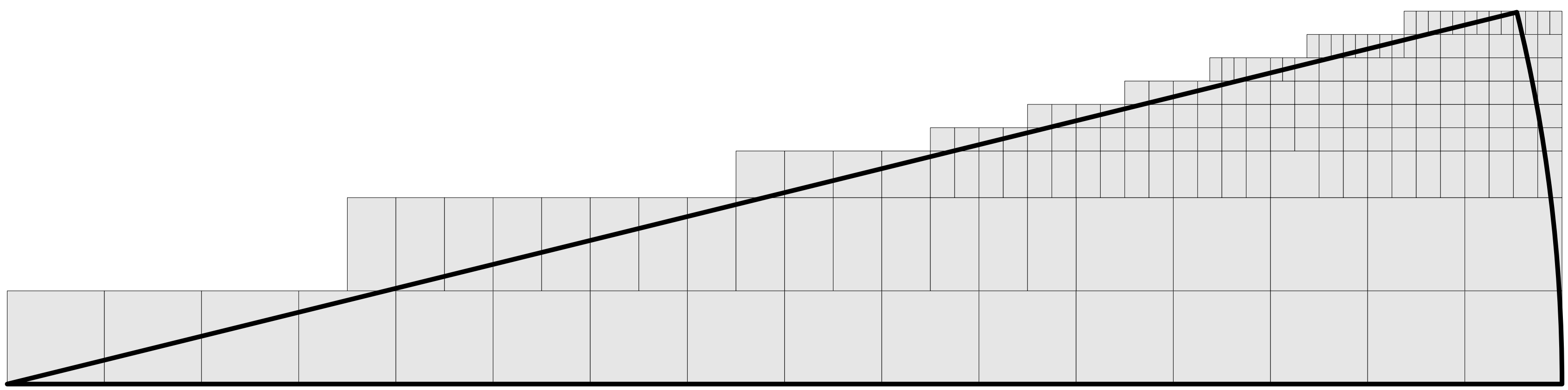


Covering for $k = 2$ (y axis)

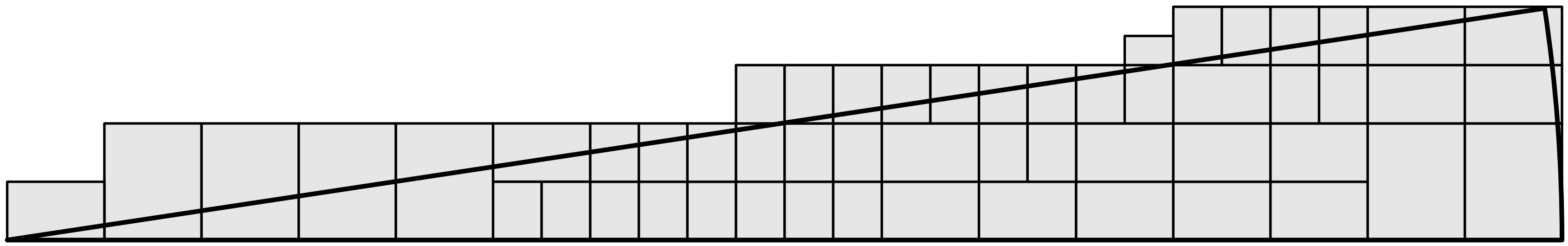


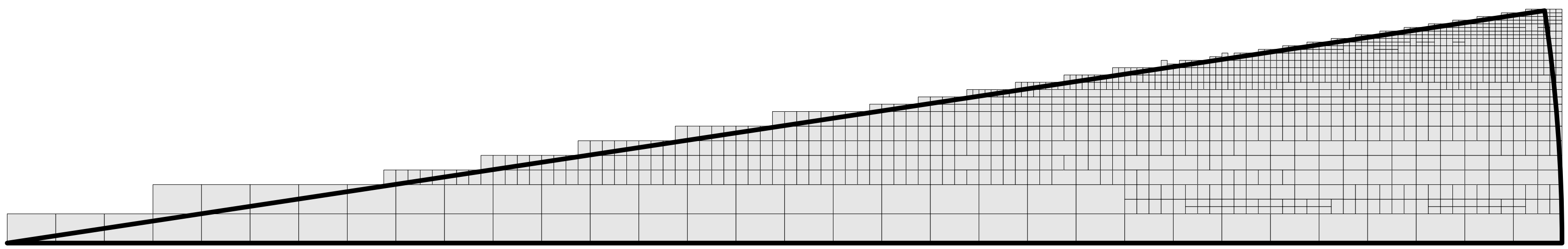
Coverings for $k = 3$





Coverings for $k = 4$





Coverings for $k = 5$