Interval Methods in Computer Graphics

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Motivation

• How do I plot an implicit curve?
  ◦ Must solve $f(x, y) = 0$
  ◦ Solution is a curve, but where is it?

• How do I render an implicit surface?
  ◦ Must solve $f(x, y, z) = 0$ for $(x, y, z)$ on a ray
  ◦ Solution is one or more points, but need point closest to eye!

• How do I intersect two parametric surfaces?
  ◦ Must solve $f(u, v) = g(s, t)$
  ◦ Solution is a set of curves in space and a set of curves in each parametric plane. Where are they? How do they match?
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Plotting an implicit curve

\[ y^2 - x^3 + x = 0 \]

\[ \Omega = [-2, 2] \times [-2, 2] \]
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Rendering implicit surfaces

\[ 4(x^4 + (y^2 + z^2)^2) + 17x^2(y^2 + z^2) - 20(x^2 + y^2 + z^2) + 17 = 0 \]
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Intersecting two parametric surfaces

(Snyder, 1992)
Interval arithmetic
Can we trust floating-point arithmetic?

Rump’s example – Evaluate this innocent-looking polynomial expression:

\[ f = 333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + x/(2y), \]

for \( x = 77617 \) and \( y = 33096 \).

\[
\begin{align*}
f &:= 333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + x/(2y); \\
f &:= 1.172603940
\end{align*}
\]

\[
\begin{align*}
f &:= 33375/100y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 55/10y^8 + x/(2y); \\
f &:= -0.8273960599
\end{align*}
\]

Not Maple’s fault! Running gcc under Linux gives \( 5.76461 \times 10^{17} \).

*Culprit is catastrophic cancellation of floating-point arithmetic!*
Interval arithmetic

- To improve reliability of floating-point computations (Moore, 1960)
- Represent quantities as intervals:
  \[ x \sim [a, b] \Rightarrow x \in [a, b] \]
- Operate with intervals generating other intervals:
  - Simple formulas for elementary operations and functions:
    \[
    \begin{align*}
    [a, b] + [c, d] &= [a + c, b + d] \\
    [a, b] \times [c, d] &= \left[\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}\right] \\
    [a, b] / [c, d] &= [a, b] \times [1/d, 1/c] \\
    [a, b]^2 &= [0, \max(a^2, b^2)] \text{ when } 0 \in [a, b] \\
    \exp [a, b] &= [\exp(a), \exp(b)] \\
    \ldots
    \end{align*}
    \]
  - Automatic extensions for complicated expressions
  - Rounding control available in modern floating-point units (IEEE 754)
Interval arithmetic

- Every expression $f$ has an interval extension $F$:
  $$x_i \in X_i \implies f(x_1, \ldots, x_n) \in F(X_1, \ldots, X_n)$$

- Interval computations not immune to roundoff errors
  Wide results alert user of catastrophic cancellation
- Roundoff errors are not our main motivation!
- Interval computations allow range estimates and avoid point sampling

  $$F(X) \supseteq f(X) = \{ f(x) : x \in X \}$$

For instance

  $$0 \not\in F(X) \implies 0 \not\in f(X) \implies f = 0 \text{ has no solution in } X$$

This is a computational proof!
Interval probing of implicit curve

\[ y^2 - x^3 + x = 0 \]

\[
\begin{align*}
X &= [-2, -1] \\
Y &= [1, 2] \\
X^3 &= [-8, -1] \\
-X^3 &= [1, 8] \\
-X^3 + X &= [-1, 7] \\
Y^2 &= [1, 4] \\
Y^2 - X^3 + X &= [0, 11]
\end{align*}
\]

- Interval estimates not tight

\[ f(X, Y) = [1, 10] \subset [0, 11] \]

- Interval estimates improve as intervals shrink
Interval probing of implicit curve

$[-2, -1] \times [1, 2]$  

$[0, 11]$  

yes?
Interval probing of implicit curve

\([-2, -1.5] \times [1.5, 2]\] \quad [3.625, 10.5] \quad \text{no}
Interval probing of implicit curve

$[-1.5, -1] \times [1.5, 2] \quad [1.75, 6.375] \quad \text{no}$
Interval probing of implicit curve

\([-2, -1.5] \times [1, 1.5]\) \hspace{1cm} [2.375, 8.75] \hspace{1cm} \text{no}
Interval probing of implicit curve

$[-1.5, -1] \times [1, 1.5]$, $[0.5, 4.625]$, no
Interval probing of implicit curve

$[-2, -1] \times [1, 2] \quad [0.5, 10.5] \quad \text{no!}$
Approximation of implicit curve
Robust adaptive enumeration

- Recursive exploration of domain Ω starts with explore(Ω)
- Discard subregions X of Ω when 0 ∉ F(X)
  = proof that X does not contain any part of the curve!

\[
\text{explore}(X):
\begin{align*}
  &\text{if } 0 \notin F(X) \text{ then} \\
  &\quad \text{discard } X \\
  &\text{elseif } \text{diam}(X) < \varepsilon \text{ then} \\
  &\quad \text{output } X \\
  &\text{else} \\
  &\quad \text{divide } X \text{ into smaller pieces } X_i \\
  &\quad \text{for each } i, \text{ explore}(X_i)
\end{align*}
\]

- Output cells have the same size: only spatial adaption

Robust adaptive approximation

- Estimate curvature by gradient variation
- $G$ = inclusion function for the normalized gradient of $f$
- $G(X)$ small $\Rightarrow$ curve approximately flat inside $X$

```
explore(X):
    if $0 \not\in F(X)$ then
discard $X$
    elseif diam($X$) $< \varepsilon$ or diam($G(X)$) $< \delta$ then
approx($X$)
    else
        divide $X$ into smaller pieces $X_i$
        for each $i$, explore($X_i$)
```

- Output cells vary in size: spatial and geometrical adaption

Robust adaptive approximation
Approximation of implicit curve
Robust adaptive approximation
Robust adaptive approximation
Robust adaptive approximation
Offsets of parametric curves

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Offsets of parametric curves
Bisectors of parametric curves
Bisectors of parametric curves
Bisectors of parametric curves
Bisectors of parametric curves
Medial axis of parametric curves
Interval methods

• Robust: they don’t lie
  ◦ correctness depends on $F(X) \supseteq f(X)$
  ◦ can prove $0 \notin f(X)$, not that $0 \in f(X)$

• Converge: solutions get better
  ◦ $F(X) \to \{f(x)\}$ as $X \to \{x\}$

• Conservative: they tend to exaggerate
  ◦ $f(x, y) = y^2 - x^3 + x$  \hspace{1cm} $X = [-2, -1] \times [1, 2]$
    \hspace{1cm} $F(X) = [0, 11]$ \hspace{1cm} $f(X) = [1, 10]$
  ◦ gets worse in complicated expressions and iterative methods

• Efficient?
  ◦ how much larger is $F(X)$?
  ◦ better estimates imply faster methods
The dependency problem in interval arithmetic

IA can’t see correlations between operands

\[ g(x) = (10 + x)(10 - x) \text{ for } x \in [-2, 2] \]

\[
\begin{align*}
10 + x &= [8, 12] \\
10 - x &= [8, 12] \\
(10 + x)(10 - x) &= [64, 144] \quad \text{diam} = 80 \\
\text{Exact range} &= [96, 100] \quad \text{diam} = 4
\end{align*}
\]
IA can’t see correlations between operands

\[ g(x) = (10 + x)(10 - x) \] for \( x \in [-u, u] \)

\[
\begin{align*}
10 + x &= [10 - u, 10 + u] \\
10 - x &= [10 - u, 10 + u] \\
(10 + x)(10 - x) &= [(10 - u)^2, (10 + u)^2] \quad \text{diam} = 40u \\
\text{Exact range} &= [100 - u^2, 100] \quad \text{diam} = u^2
\end{align*}
\]
The dependency problem in interval arithmetic

\[ g(x) = \frac{\sqrt{x^2 - x + 1/2}}{\sqrt{x^2 + 1/2}} \]

\[ g^n \rightarrow c = \text{fixed point of } g \approx 0.5586, \text{ but intervals diverge} \]

Interval estimates may get too large in long computations
Affine arithmetic
Affine arithmetic

AA represents a quantity $x$ with an affine form

$$\hat{x} = x_0 + x_1 \varepsilon_1 + \cdots + x_n \varepsilon_n$$

- Noise symbols $\varepsilon_i \in [-1, +1]$: independent, but otherwise unknown
- Can compute arbitrary formulas on affine forms
  - Need affine approximations for non-affine operations
  - New noise symbols created during computation due to approximation and rounding
- Can replace IA
  - $x \sim \hat{x} \Rightarrow x \in [x_0 - r, x_0 + r]$ for $r = |x_1| + \cdots + |x_n|$  
  - $x \in [a, b] \Rightarrow x \sim \hat{x} = x_0 + x_1 \varepsilon_1$
    - $x_0 = (b + a)/2, \quad x_1 = (b - a)/2$
The dependency problem in interval arithmetic – AA version

AA can see correlations between operands

\[ g(x) = (10 + x)(10 - x) \] for \( x \in [-u, u] \), \( x = 0 + u \varepsilon \)

\[
\begin{align*}
10 + x &= 10 - u \varepsilon \\
10 - x &= 10 + u \varepsilon \\
(10 + x)(10 - x) &= 100 - u^2 \varepsilon \\
\text{range} &= [100 - u^2, 100 + u^2] \quad \text{diam} = 2u^2 \\
\text{Exact range} &= [100 - u^2, 100] \quad \text{diam} = u^2
\end{align*}
\]
AA can see correlations between operands

\[ g(x) = (10 + x)(10 - x) \text{ for } x \in [-u, u], \quad x = 0 + u \varepsilon \]

\[
\begin{align*}
10 + x & = 10 - u \varepsilon \\
10 - x & = 10 + u \varepsilon \\
(10 + x)(10 - x) & = 100 - u^2 \varepsilon \\
\text{range} & = [100 - u^2, 100 + u^2] \quad \text{diam} = 2u^2 \\
\text{Exact range} & = [100 - u^2, 100] \quad \text{diam} = u^2
\end{align*}
\]
The dependency problem in interval arithmetic – AA version

\[ g(x) = \sqrt{x^2 - x + 1/2} / \sqrt{x^2 + 1/2} \]
Replacing IA with AA for plotting implicit curves

\[ x^2 + y^2 + xy - \frac{(xy)^2}{2} - \frac{1}{4} = 0 \]
Replacing IA with AA for surface intersection

Tensor product Bézier surfaces of degree \((p, q)\):

\[
f(u, v) = \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij} B_i^p(u) B_j^q(v), \quad B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}\n\]
Replacing IA with AA for surface intersection

(Figueiredo, 1996)
Exploiting the correlations given by AA
Affine forms that share noise symbols are not independent:

\[ \hat{x} = x_0 + x_1 \varepsilon_1 + \cdots + x_n \varepsilon_n \]
\[ \hat{y} = y_0 + y_1 \varepsilon_1 + \cdots + y_n \varepsilon_n \]

The region containing \((x, y)\) is

\[ Z = \{(x, y) : \varepsilon_i \in U\} \]

This region is the image of \(U^n\) under an affine map \(\mathbb{R}^n \rightarrow \mathbb{R}^2\). It’s a centrally symmetric convex polygon, a zonotope.
Geometry of affine arithmetic

Affine forms that share noise symbols are not independent:

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Z = \{(x, y) : \epsilon_i \in U\}
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This region is the image of \(U^n\) under an affine map \(\mathbb{R}^n \rightarrow \mathbb{R}^2\). It’s a centrally symmetric convex polygon, a \textit{zonotope}.

The region would be a rectangle if \(x\) and \(y\) were independent.
Given a parametric curve $C = \gamma(I)$, where $\gamma: I \rightarrow \mathbb{R}^2$ and $T \subseteq I$, compute a bounding rectangle for $\mathcal{P} = \gamma(T)$. 
Approximating parametric curves

Given a parametric curve $C = \gamma(I)$, where $\gamma: I \rightarrow \mathbb{R}^2$ and $T \subseteq I$, compute a bounding rectangle for $\mathcal{P} = \gamma(T)$. 
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Approximating parametric curves

Given a parametric curve $C = \gamma(I)$, where $\gamma: I \to \mathbb{R}^2$ and $T \subseteq I$, compute a bounding rectangle for $P = \gamma(T)$.

Solution:

- Write $\gamma(t) = (x(t), y(t))$.
- Represent $t \in T$ with an affine form:
  \[ \hat{t} = t_0 + t_1 \varepsilon_1, \quad t_0 = (b + a)/2, \quad t_1 = (b - a)/2 \]
- Compute coordinate functions $x$ and $y$ at $\hat{t}$ using AA:
  \[ \hat{x} = x_0 + x_1 \varepsilon_1 + \cdots + x_n \varepsilon_n \]
  \[ \hat{y} = y_0 + y_1 \varepsilon_1 + \cdots + y_n \varepsilon_n \]
- Use bounding rectangle of the $xy$ zonotope.
Approximating parametric curves
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Approximating parametric curves

(Figueiredo–Stolfi –Velho, 2003)
Ray casting implicit surfaces

- Implicit surface
  \[ h : \mathbb{R}^3 \to \mathbb{R} \]
  \[ S = \{ p \in \mathbb{R}^3 : h(p) = 0 \} \]

- Ray
  \[ r(t) = E + t \cdot v, \quad t \in [0, \infty) \]

- Ray intersects \( S \) when
  \[ f(t) = h(r(t)) = 0 \]

- First intersection occurs at *smallest* zero of \( f \) in \([0, \infty)\).

- Paint pixel with color based on normal at first intersection point
Ray casting implicit surfaces

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\[
4(x^4 + (y^2 + z^2)^2) + 17x^2(y^2 + z^2) - 20(x^2 + y^2 + z^2) + 17 = 0
\]
(Custatis–Figueiredo–Gattass, 1999)
Interval bisection

- Solve $f(t) = 0$ using inclusion function $F$ for $f$:
  \[ F(T) \supseteq f(T) = \{ f(t) : t \in T \}, \quad T \subseteq I \]
- $0 \notin F(T) \Rightarrow$ no solutions of $f(t) = 0$ in $T$
- $0 \in F(T) \Rightarrow$ there may be solutions in $T$

interval-bisection([a, b]):
  if $0 \in F([a, b])$ then
    $c \leftarrow (a + b)/2$
    if $(b - a) < \varepsilon$ then
      return $c$
    else
      interval-bisection([a, c]) ← try left half first!
      interval-bisection([c, b])

Start with interval-bisection([0, $t_\infty$]) to find the first zero.
Ray casting implicit surfaces with affine arithmetic

- AA exploits linear correlations of $x, y, z$ in $f(t) = h(r(t))$
- AA provides additional information
  - root must lie in smaller interval
  - quadratic convergence near simple zeros
Sampling procedural shaders

IA

AA (Heidrich–Slusallek–Seidel)
Conclusion

Interval methods have a place for solving computer graphics problems:

- Give reliable way to probe the global behavior of functions
- Lead naturally to robust, adaptive algorithms
- Several good libraries available on the internet

Affine arithmetic is a useful tool for interval methods

- AA more accurate than IA
- AA provides additional information that can be exploited
- AA locally more expensive than IA but globally more efficient
- AA has geometric flavor

Lots more to be done!
Some references

  general philosophy
- Toth, *SIGGRAPH*, 1985
  ray tracing parametric surfaces
  ray tracing implicit surfaces
  interval methods in computer graphics
  three applications
Some references

- Duff, *SIGGRAPH*, 1992
  implicit functions and constructive solid geometry
- Snyder, *SIGGRAPH*, 1992; also book
  interval analysis for computer graphics
  intersection of parametric surfaces
  ray tracing parametric surfaces
  sampling procedural shaders
- Tupper, MSc thesis, Toronto, 1996; also *SIGGRAPH*, 2001
  plotting implicit relations with GraphEq