Images of Julia sets that you can trust

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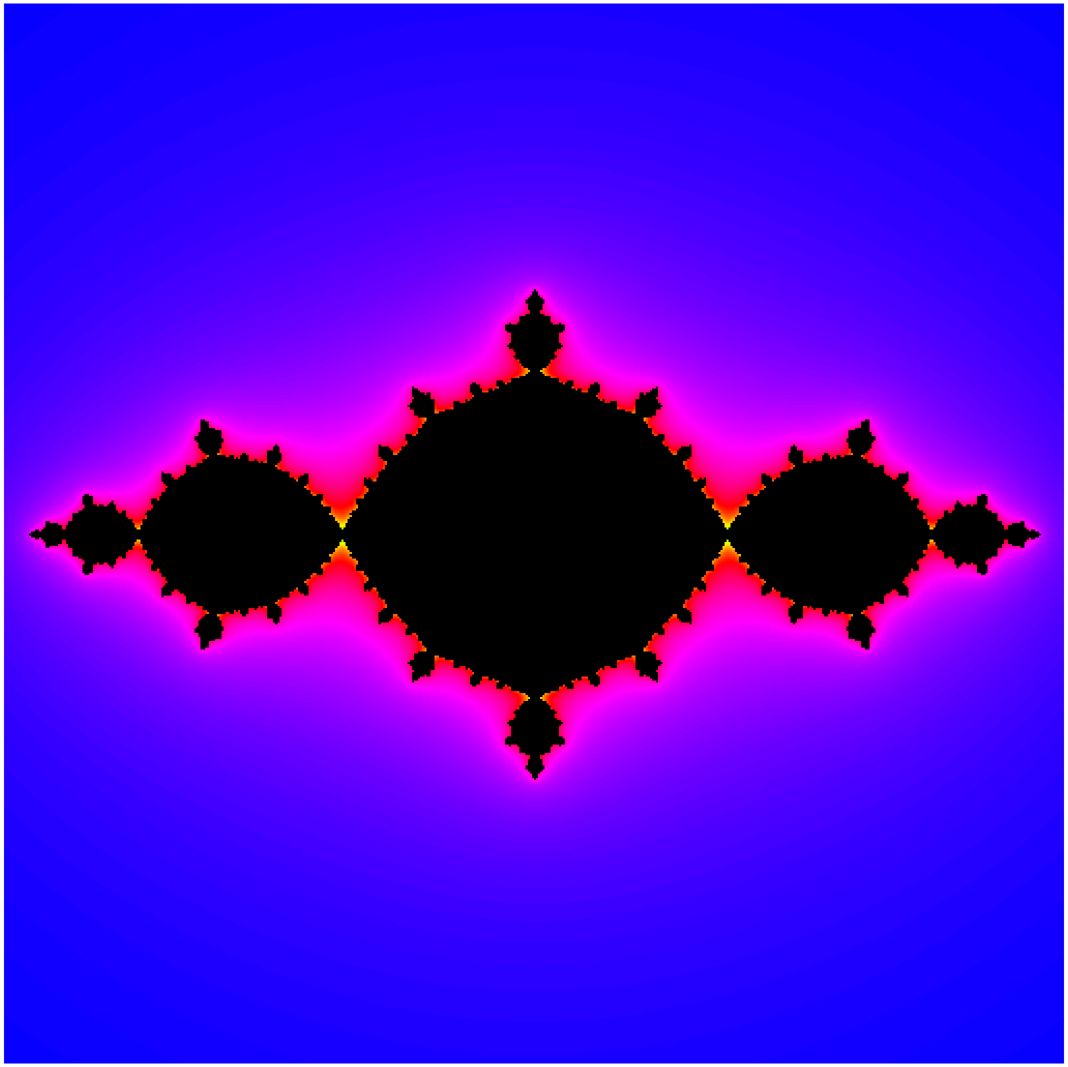
Can we trust this beautiful image?



Julia sets

Let $f : \mathbf{C} \to \mathbf{C}$, $f(z) = z^2 + c$, where $c \in \mathbf{C}$ is fixed.

What is the *dynamics* of f?

What happens with the the *orbit* of $z_0 \in \mathbf{C}$ under f?

$$z_1 = f(z_0), \quad z_2 = f(z_1), \quad \ldots, \quad z_n = f(z_{n-1}) = f^{(n)}(z_0)$$

Some orbits stay bounded forever. Other orbits go away to infinity.

Attraction basin of ∞	$A(\infty) = \{z_0 \in \mathbf{C} : \lvert f^{(n)}(z_0)\rvert \to \infty\}$
Julia set of f	$J = \partial A(\infty)$
Filled Julia set of f	$K = \mathbf{C} \setminus A(\infty)$

The Julia set is usually a fractal and so is elusive to draw.

Pictures usually show the filled Julia set instead.

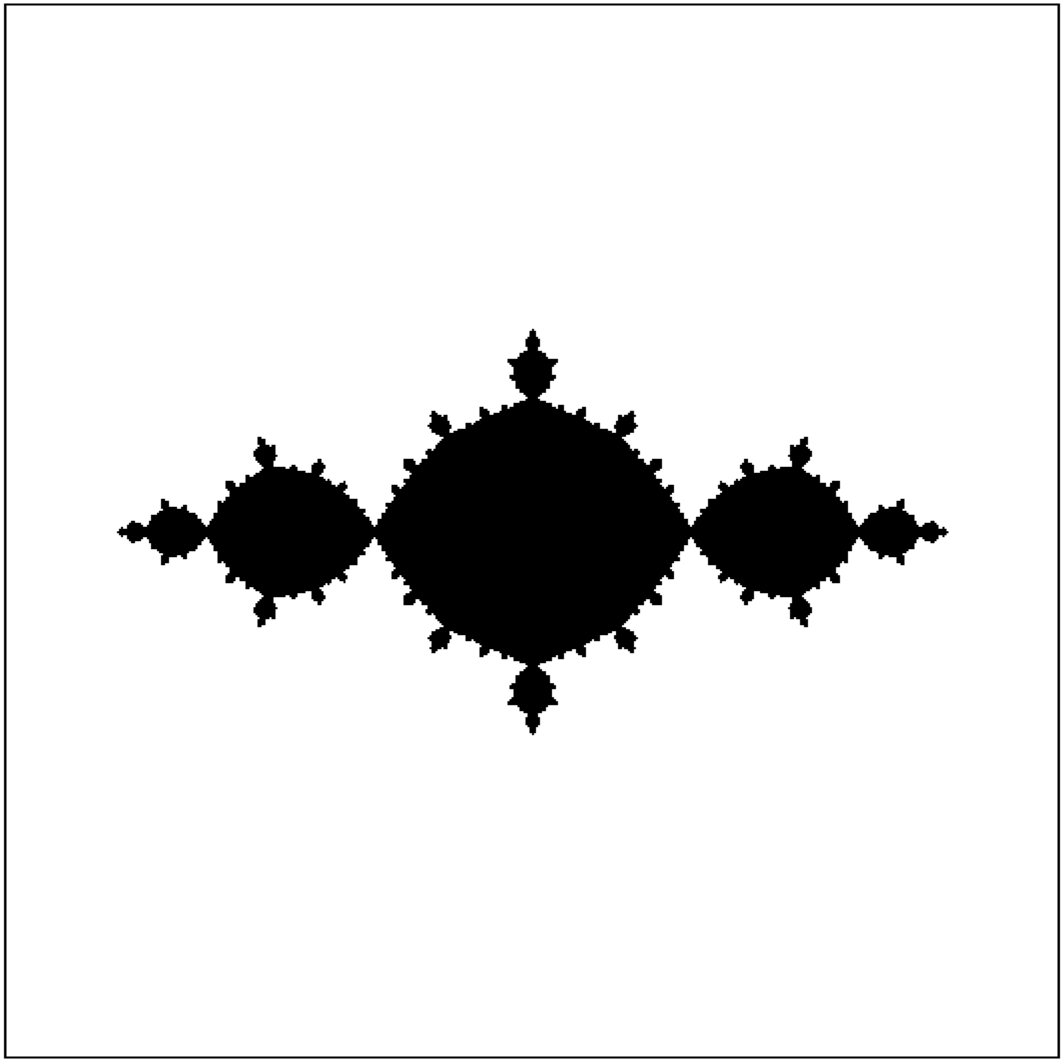
Popular algorithm for generating images of Julia sets

Crucial observation: If an orbit ever goes outside the circle B of radius $R = \max(|c|, 2)$ centered at the origin, then it goes away to infinity.

Simple algorithm for drawing the filled Julia set K in a region $\Omega \subset \mathbf{C}$:

- Choose a large integer N.
- Lay a grid of pixels over Ω.
- For each pixel in the image:
  - ◇ Compute up to N points of the orbit starting at the pixel center.
  - ◇ If the orbit goes outside B, then paint the pixel white.
  - ◇ If the orbit remains inside B, then paint the pixel black.
- K is the black region, $A(\infty)$ is the white region.

Typical image computed with popular algorithm



No guarantees given:

What happens between pixels?

What happens for larger N?

What about round-off?

White pixels hint at $A(\infty)$

Black pixels may turn white

Border pixels uncertain

Tools for computing guaranteed images of Julia sets

The main tool is interval arithmetic.

Extend $f(z) = z^2 + c$ to F defined on rectangles $Z \subset \mathbf{C}$

$$F(Z) \supseteq f(Z) = \{f(z) : z \in Z\}$$

Then $F^{(m)}(Z) \supseteq f^{(m)}(Z)$ for all $m \in \mathbf{N}$.

Validating the exterior of K:

If $F^{(m)}(Z)$ is outside B for some m, then *all* orbits starting in Z are unbounded, and so $Z \subseteq A(\infty)$.

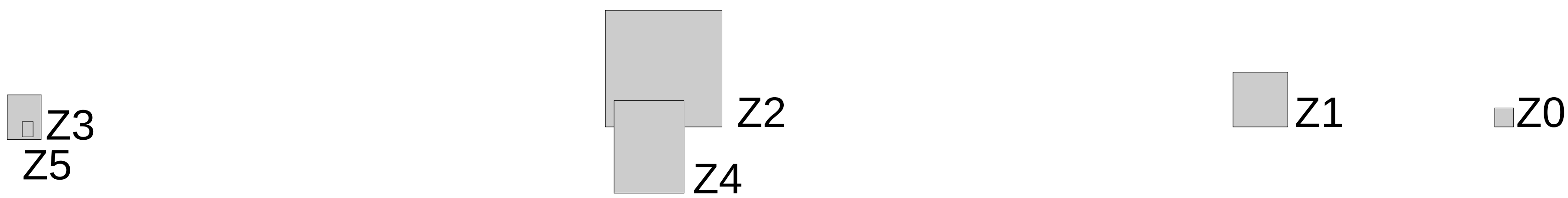
Validating the interior of K:

If $F^{(m)}(Z) \subseteq F^{(m_0)}(Z)$ for $m > m_0$ and $F^{(k)}(Z) \subseteq B$ for all $k \leq m$, then $F^{(k)}(Z) \subseteq B$ for all $k \in \mathbf{N}$, and so $Z \subseteq K$.

These are *computational proofs*!

Validating the interior of K – example

$c = -1, \quad Z_k = F^{(k)}(Z), \quad Z = [1.40625, 1.43750] \times [0, 0.03125]$



$Z_0 \subseteq B, \ldots, Z_5 \subseteq B$ and $Z_5 \subseteq Z_3 \Rightarrow$ *all* orbits starting at Z remain inside $Z_3 \cup Z_4$ and so $Z \subseteq K$.

In general, check whether Z_k is inside the $Z_0 \cup \ldots \cup Z_{k-1} \subseteq B$, or even inside the union of *previously validated* rectangles.

Algorithm that computes guaranteed images

Recursive, adaptive algorithm:

```
explore(Z):
    status ← orbit(Z)
    if status = "unbounded"
        paint Z white
    elseif status = "bounded"
        paint Z black
    elseif diam(Z) ≤ ε then
        paint Z grey
    else
        split Z into Z_1, Z_2, Z_3, Z_4
        explore(Z_j) for j = 1, 2, 3, 4
```

Start with explore(Ω)

Guarantees:

All points in the white region have unbounded orbits.

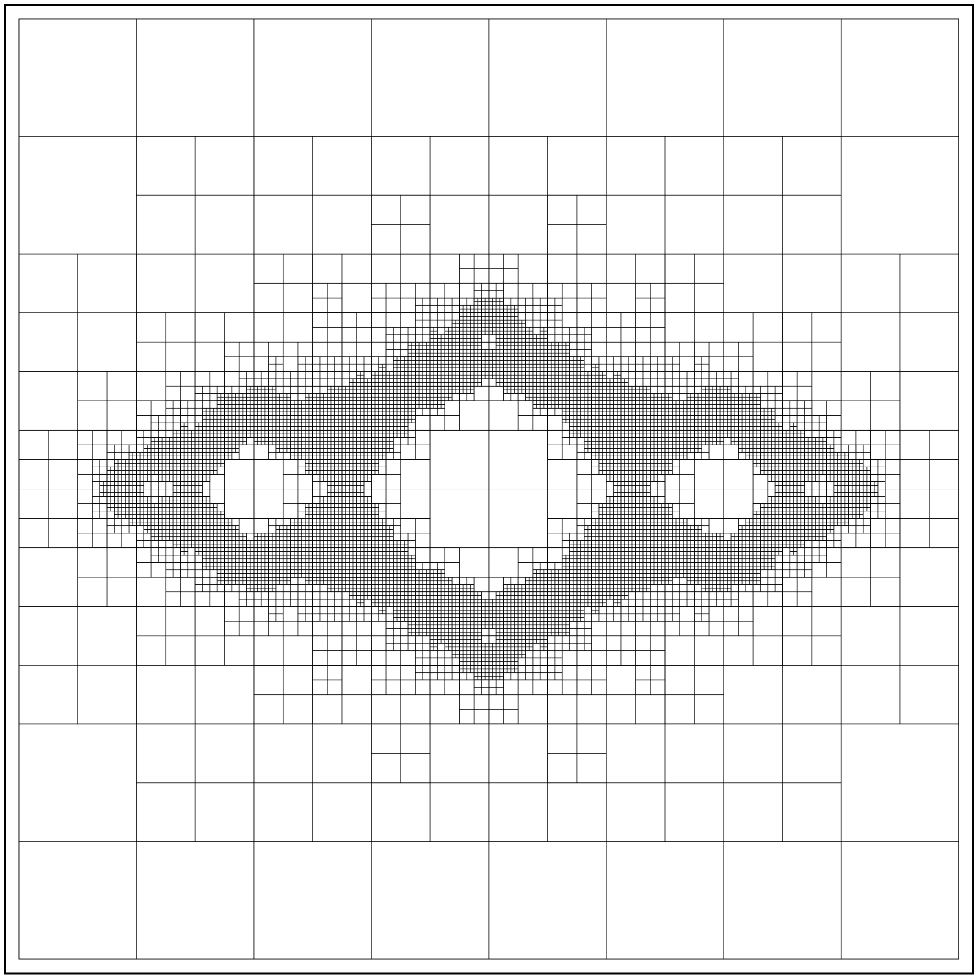
All points in the black region have bounded orbits.

K is definitely inside the union of the black and grey regions.

J is definitely inside the grey region.

No sampling!

Algorithm that computes guaranteed images



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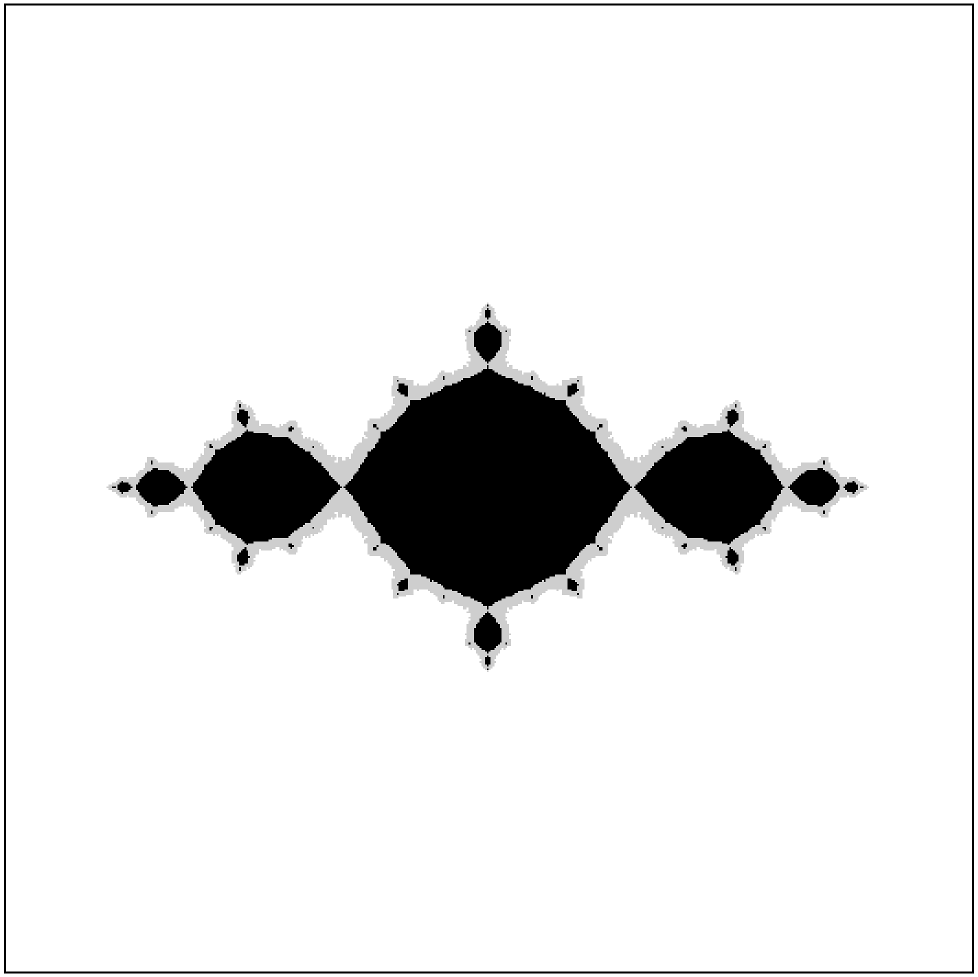

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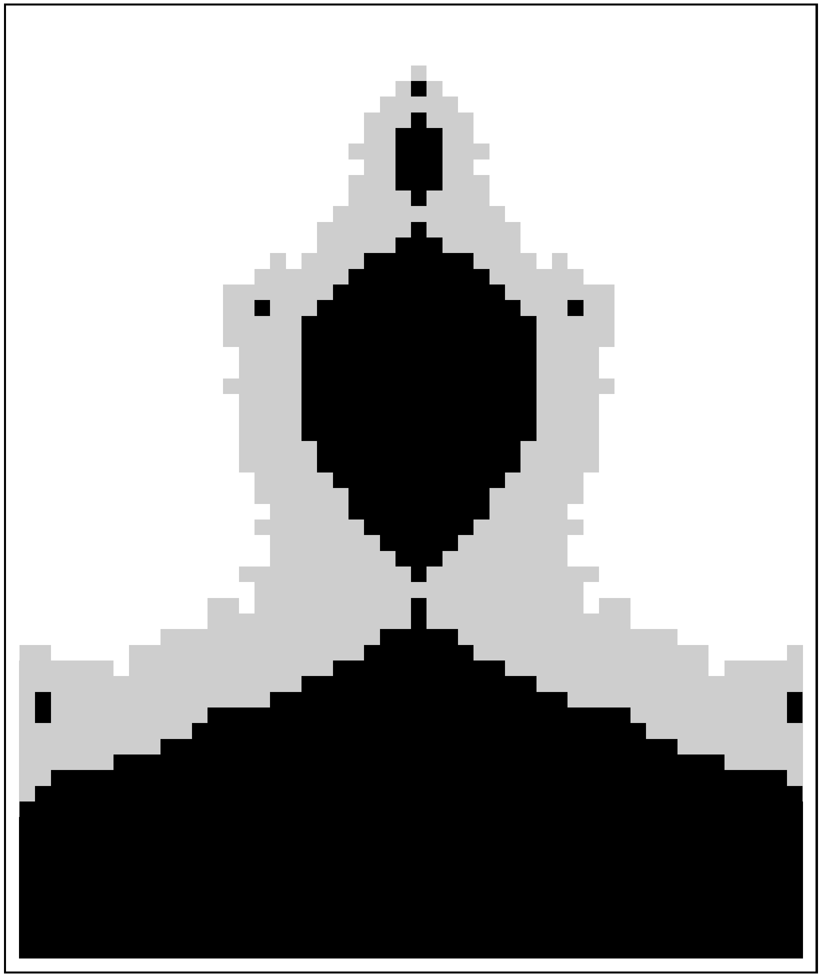

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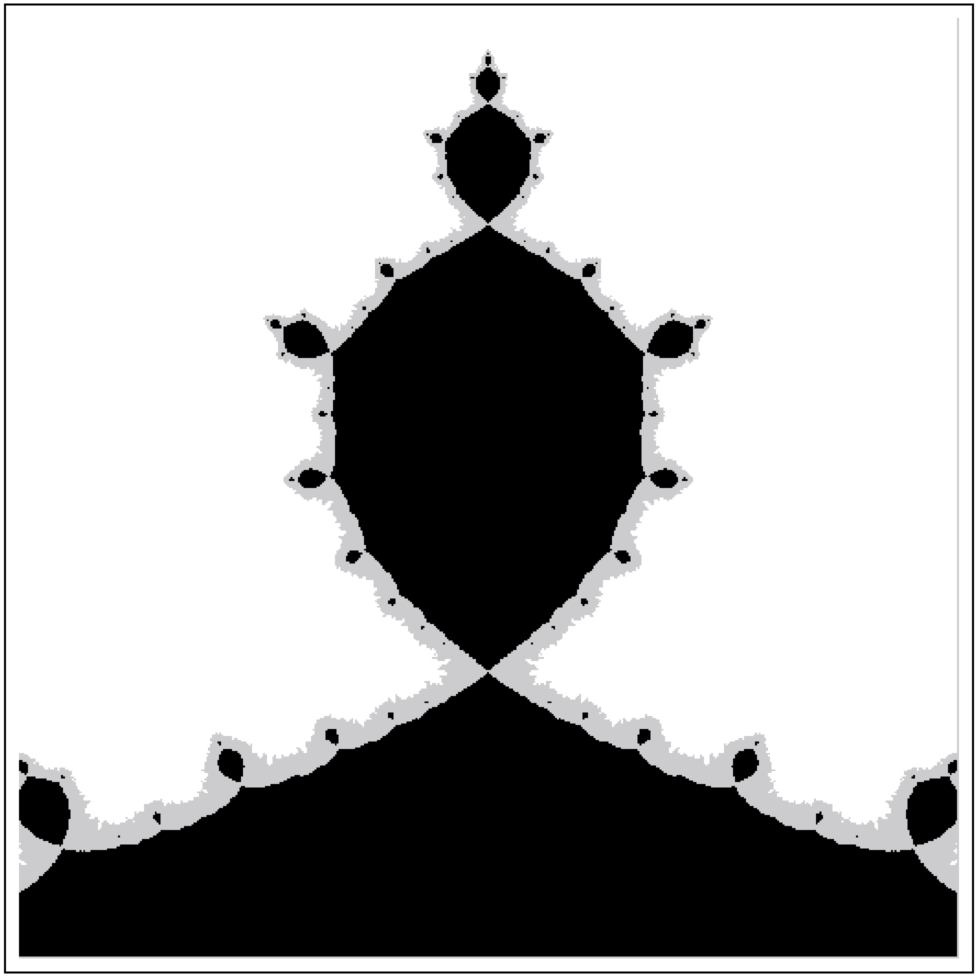

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Are these the first verified pictures of Julia sets?